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The bosonic mother of fermionic D-branes

Auttakit Chattaraputi,[a] François Englert,[b] Laurent Houart[c*] and Anne Taormina[a]

[a] *Department of Mathematical Sciences, University of Durham South Road, DH1 3LE Durham, England*
[b] *Service de Physique Théorique, Université Libre de Bruxelles Campus Plaine, C.P.225, Boulevard du Triomphe, B-1050 Bruxelles, Belgium*
[c] *Service de Physique Théorique et Mathématique Université Libre de Bruxelles Campus Plaine C.P. 231, Boulevard du Triomphe, B-1050 Bruxelles, Belgium*
E-mail: auttakit.chattaraputi@durham.ac.uk, anne.taormina@durham.ac.uk, fenglert@ulb.ac.be, lhouart@ulb.ac.be

Abstract: We extend the search for fermionic subspaces of the bosonic string compactified on $E_8 \times \mathrm{SO}(16)$ lattices to include *all* fermionic D-branes. This extension constraints the truncation procedure previously proposed and relates the fermionic strings, supersymmetric or not, to the global structure of the SO(16) group. The specific properties of all the fermionic D-branes are found to be encoded in its universal covering, whose maximal toroid defines the configuration space torus of their mother bosonic theory.

Keywords: Bosonic Strings, D-branes, Superstring Vacua.

[*] Research Associate F.N.R.S.

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Contents

1. Introduction and discussion

It is well-known that ten-dimensional fermionic strings can be analysed in terms of bosonic operators, a consequence of the boson-fermion equivalence in two dimensions. The approach taken here is different. We wish to generalize previous considerations whereby the Hilbert space of the perturbative fermionic closed strings were obtained as subspaces of the 26-dimensional closed bosonic string theory compactified on suitable 16-dimensional manifolds [1, 2]. Truncating the bosonic string Hilbert space to its fermionic subspaces yields the conformal field theories of these fermionic strings.

The extension of the truncation procedure to open bosonic strings poses a problem. Constraints arise because the holomorphic and antiholomorphic conformal algebras describing the closed string sectors of the bosonic theory are related at the conformal boundary accommodating open strings. These constraints are easily taken into account when the torus partition function is diagonal, and, in reference [3], tadpole-free open descendants[1] of type IIB and type OB were obtained in this way from bosonic parents. Many properties of these open fermionic strings, such as tadpoles and anomaly cancellations, Chan-Paton groups, D-brane and orientifold tensions, were found to be encoded in the bosonic string and were derived in a simple way from bosonic considerations alone.

[1]For a comprehensive review on open strings see reference [4].

In order to get *all* open sectors of fermionic strings, bosonic parents with non-diagonal torus partition functions must be introduced. The constraints have then non-trivial consequences. We shall find that they tightly relate all bosonic parents to each other as well as to their fermionic offsprings.

Let us first recall how a relation between bosonic and fermionic closed string theories first suggested by Freund [5] was established [1, 2], and how it was extended to open strings for diagonal conformal field theories [3].

The basic issue was to uncover space-time fermions from truncation of a compactified bosonic string. This was achieved in the following way:

1. One compactifies, in the light-cone gauge, $d = 16$ transverse dimensions on a torus, leaving eight non-compact dimensions with transverse group[2] $\mathrm{SO}_{\mathrm{trans}}(8)$. One chooses a compactification of the closed string at an enhanced symmetry point with gauge group $\mathcal{G}_L \times \mathcal{G}_R$ where $\mathcal{G}_L$ and $\mathcal{G}_R$ are semi-simple simply laced Lie groups. Recall that in terms of the left and right compactified momenta, the mass spectrum is
$$\frac{\alpha' m_L^2}{4} = \alpha' \mathbf{p_L}^2 + N_L - 1\,,$$
$$\frac{\alpha' m_R^2}{4} = \alpha' \mathbf{p_R}^2 + N_R - 1\,, \tag{1.1}$$
and
$$m^2 = \frac{m_L^2}{2} + \frac{m_R^2}{2}\,; \qquad m_L^2 = m_R^2\,. \tag{1.2}$$
In eq. (1.1) N_L and N_R are the oscillator numbers in 26-dimensions and the zero-modes $\sqrt{2\alpha'}\mathbf{p_L}$, $\sqrt{2\alpha'}\mathbf{p_R}$, where α' is the string length squared, span a $2d$-dimensional even self-dual lorentzian lattice with negative (resp. positive) signature for left (resp. right) momenta. This ensures modular invariance of the closed string spectrum [6]. For generic toroidal compactifications, the massless vectors $\alpha^\mu_{-1,R}\ \alpha^i_{-1,L}|0_L, 0_R\rangle$ and $\alpha^\mu_{-1,L}\ \alpha^i_{-1,R}|0_L, 0_R\rangle$, where the indices μ and i respectively refer to non-compact and compact dimensions, generate a local symmetry $[U_L(1)]^d \times [U_R(1)]^d$. But more massless vectors arise when $\sqrt{2\alpha'}\mathbf{p_L}$ and $\sqrt{2\alpha'}\mathbf{p_R}$ are roots of simply laced groups $\mathcal{G}_L$ and $\mathcal{G}_R$ of rank d (with root length $\sqrt{2}$). The gauge symmetry is enlarged to $\mathcal{G}_L \times \mathcal{G}_R$.

   For closed strings, the compactification lattice in both sectors (or in the right sector only for the heterotic strings) is taken to be a sublattice of the $E_8 \times \mathrm{SO}(16)$ weight lattice. This sublattice must preserve modular invariance, which means that the left and right compactified momenta $\sqrt{2\alpha'}\mathbf{p_L}$, $\sqrt{2\alpha'}\mathbf{p_R}$ must span a $2d$-dimensional even self-dual lorentzian lattice.

2. This type of compactification produces gauged symmetries in subgroups $\mathrm{SO}_{\mathrm{int}}(8)$ of $\mathrm{SO}(16)$ in both sectors (or in the right sector only for the heterotic strings). The $\mathrm{SO}_{\mathrm{int}}(8)$ groups are then mapped onto $\mathrm{SO}_{\mathrm{trans}}(8)$ in such a way that the diagonal

[2] Throughout the paper, any group locally isomorphic to the rotational group of dimension $n(n-1)/2$ will be labelled $\mathrm{SO}(n)$. When specifically referring to the universal covering group, we shall use the notation $\widetilde{\mathrm{SO}}(n)$.

algebra $\mathrm{so}_{\mathrm{diag}}(8) = \mathrm{diag}[\mathrm{so}_{\mathrm{trans}}(8) \times \mathrm{so}_{\mathrm{int}}(8)]$ becomes identified with a new transverse algebra. The spinor representations of $\mathrm{SO}_{\mathrm{int}}(8)$ describe fermionic states because a rotation in space induces a half-angle rotation on these states. The consistency of the above procedure relies on the possibility of extending the diagonal algebra $\mathrm{so}_{\mathrm{diag}}(8)$ to the new full Lorentz algebra $\mathrm{so}_{\mathrm{diag}}(9,1)$, a highly non-trivial constraint. To break the original Lorentz group $\mathrm{SO}(25,1)$ in favour of the new one, a truncation consistent with conformal invariance must be performed on the physical spectrum of the bosonic string. Actually, the states described by twelve compactified bosonic fields must be projected out, except for momentum zero-modes of unit length [1, 2]. The removal of twelve bosonic fields accounts for the difference between the bosonic and fermionic light cone gauge central charges. Namely, in units where the central charge of a boson is one, this difference counts 11 for the superghosts and (1/2).2 for time-like and longitudinal Majorana fermions. The zero-modes of length $\ell = 1$ kept in the twelve truncated dimensions contribute a constant $\ell^2/2$ to the mass. They account for the removal by truncation of the oscillator zero-point energies in these dimensions, namely for $-(-1/24) \cdot 12 = +1/2$.

3. This procedure can be extended to stable open strings descendants of the type IIB and type OB theories [3]. Requiring the absence of tadpole divergencies in the bosonic string , the correct Chan-Paton groups for the tadpole-free fermionic strings, including the anomaly-free SO(32) group of type I were obtained by truncation. In addition, the tensions of space-filling D-branes and of orientifolds in the fermionic theories followed from the tensions of their bosonic parents.

To extend the analysis to all fermionic Dp-branes and to open string descendants of type IIA and type OA theories, we introduce bosonic parents with non-diagonal torus partition functions. All the parent theories, including the diagonal theories from which the above results were obtained, are related to each other and to the fermionic theories living in their fermionic subspaces through the global properties of SO(16). Compactification of the bosonic string on the configuration space torus defined by the maximal abelian subgroup (maximal toroid) $\widetilde{\mathcal{T}}$ of the $E_8 \times \widetilde{\mathrm{SO}}(16)$ group yields the 'mother theory'[3] OB_b whose fermionic subspace is the Hilbert space of the non-supersymmetric OB theory. Subtori corresponding to all distinct maximal toroids $\widetilde{\mathcal{T}}/Z_c$ of the groups locally isomorphic to $\widetilde{\mathrm{SO}}(16)$, where Z_c labels the distinct subgroups of the centre of $\widetilde{\mathrm{SO}}(16)$, yield the parents of all the non-heterotic fermionic strings. The toroid $\widetilde{\mathcal{T}}$ encodes the information about the specific properties of *all* stable and unstable space-filling fermionic D9-branes, *all* lower dimensional Dp-branes and *all* tadpole-free descendants.

In section 2, all bosonic theories compactified on $E_8 \times \mathrm{SO}(16)$ lattices, consistent with the boundary constraints, are obtained. Their fermionic subspaces in the closed string sector are obtained from a universal truncation procedure. For IIB_b and OB_b one recovers the lattice compactifications previously obtained [3]. For IIA_b and OA_b this is not the case and new lattice compactifications do appear. In section 3, the lattice compactifications

[3]Here and in the rest of the paper the subscript b designates the bosonic parent theory.

of the bosonic string theories IIB_b, OA_b and IIA_b are deduced from the OB_b torus by identifying weight lattice points. All fermionic space-filling D-branes, stable and unstable, are obtained by truncation of the open sectors. Their characteristic properties, namely the partition functions of open strings ending on a D-brane or joining distinct D-branes, and in particular their supersymmetric or non-supersymmetric character, their charge conjugation properties and their tensions, are related to the global group properties of the 'bosonic' group $\widetilde{\mathrm{SO}}(16)$. In section 4, we confront the puzzling problem posed by the identification of the fermionic subspaces describing even dimensional fermionic branes which, in the fermionic theories, are related to the odd ones by T-dualities flipping the chirality of space-time fermions. This problem is solved in section 5. The crucial element underlying the solution is the existence of 'odd' E-dualities which relate bosonic theories differing by 'chiral' elements of Z_c. In section 6, we extend the analysis of the tadpole-free descendants given in reference [3] to obtain the OA orientifold and tadpole-free Chan-Paton group [4]. Mathematical developments are relegated to the appendices.

The identification of all the fermionic subspaces of the bosonic strings given here, both for closed and open string sectors, establishes a dictionary translating conformal field theory properties of all the fermionic strings to those of the compactified bosonic string. We cannot, at this stage, assert that the very existence of this dictionary signals a dynamical mechanism whereby the fermionic subspaces would separate from the remaining states at the non perturbative level. However, as previously pointed out [1, 2], the analogy of the stringy construction of diagonal subgroups $\mathrm{so}_{\mathrm{diag}}(8)$ to obtain space-time fermions out of bosons with the dynamical formation of fermionic monopole states in bosonic field theory [7], may perhaps be indicative of some similar dynamics. Such dynamics would of course be of fundamental importance, as it would considerably enlarge the scope of the present search for an elusive M-theory. Its quest deserves further investigations.

2. The universal symmetric truncation

In this section we construct the fermionic subspaces of the compactified closed bosonic string. We focus our attention on a given sector, say the right one.

Consider the closed bosonic string compactified on a 16-dimensional torus at an enhanced symmetry point with gauge group $\mathcal{G}_L \times \mathcal{G}_R$ and $\mathcal{G}_R = E_8 \times \mathrm{SO}(16)$.

We recall the structure of the weight lattice of the SO(16) and E_8 groups. Generically, the weight lattice of $\mathrm{SO}(4m)$ is partitioned into four cosets with respect to the $2m$-dimensional root lattice. These are the conjugacy classes $(o)_{4m}$, $(v)_{4m}$, $(s)_{4m}$, $(c)_{4m}$ which, under addition, are isomorphic to the centre $Z_2 \times Z_2$ of the universal covering group $\widetilde{\mathrm{SO}}(4m)$. The $(o)_{4m}$ lattice is the root lattice generated by the simple roots. It contains the neutral element $(0, 0, \ldots, 0)$. The $(v)_{4m}$ lattice is the vector lattice whose smallest weights are $4m$ vectors of norm one; in an orthonormal basis, these are $\sqrt{2\alpha'}\mathbf{p_v} = (\pm 1, 0, \ldots, 0)$ and permutations. The $(s)_{4m}$ and $(c)_{4m}$ lattices are spinor lattices whose smallest weights have norm squared $m/2$ and are the 2^{2m} vectors $\sqrt{2\alpha'}\mathbf{p_{s,c}} = (\pm 1/2, \pm 1/2, \ldots, \pm 1/2)$ with even (for class $(s)_{4m}$) or odd (for class $(c)_{4m}$) number of minus signs. The E_8 lattice is self-dual and its root lattice $(o)_{E_8}$ may be expressed as $(o)_{16} + (s)_{16}$.

The closed string partition function is modular invariant. Its $\mathcal{G}_L \times \mathcal{G}_R$ lattice contribution $P(\tau, \bar{\tau})$ is separately modular invariant and given by

$$P(\tau, \bar{\tau}) = \sum_{\alpha,\beta} N_{\alpha\beta} \bar{\alpha}_L(\bar{\tau}) \beta_R(\tau) , \tag{2.1}$$

where

$$\beta_R(\tau) = \sum_{\sqrt{2\alpha'}\mathbf{p}_{oR} \in (o)} \exp\left\{2\pi i\tau \left[\alpha'(\mathbf{p}_{oR} + \mathbf{p}_{\beta R})^2 + N_R^{(c)} - \frac{\delta}{24}\right]\right\} . \tag{2.2}$$

Here β is a partition function for a sublattice (β) of the $\mathcal{G}_R = E_8 \times \mathrm{SO}(16)$ weight lattice (i.e. $(\beta) = (o)_{E_8} \oplus (i)_{16}$, $i = o, v, s, c$) and $\mathbf{p}_{\beta R}$ is a fixed vector, arbitrarily chosen, of the sublattice (β). $N_R^{(c)}$ is the oscillator number in the $\delta = 16$ compact dimensions. A similar expression holds for $\bar{\alpha}_L(\bar{\tau})$, $\bar{\alpha}$ labeling a partition function for a sublattice of the weight lattice of $\mathcal{G}_L$. The coefficients $N_{\alpha\beta}$ are 0 or 1 and are chosen in such a way that $P(\tau, \bar{\tau})$ is modular invariant.

We now perform the truncation in the right sector. We decompose the SO(16) factor of $\mathcal{G}_R$ in $\mathrm{SO}'(8) \times \mathrm{SO}(8)$ and first truncate all states created by oscillators in the 12 dimensions defined by the $E_8 \times \mathrm{SO}'(8)$ root lattice. We then identify the group SO(8) with the internal symmetry group $\mathrm{SO}(8)_{\mathrm{int}}$ defined in the introduction. As pointed out there, the closure of the new Lorentz algebra dictates we keep zero-modes in the 16 compact dimensions in such a way that

$$\alpha' \mathbf{p_R}^2 [E_8 \times \mathrm{SO}(16)] = \alpha' \mathbf{p_R}^2 [\mathrm{SO}(8)] + \frac{1}{2} . \tag{2.3}$$

The zero-mode contribution $\frac{1}{2}$ in eq. (2.3) comes from $\mathrm{SO}'(8)$ as there are no vectors of norm squared one in E_8.

The decomposition of an SO(16) lattice in terms of $\mathrm{SO}'(8) \times \mathrm{SO}(8)$ lattices yields

$$\begin{aligned}
(o)_{16} &= [(o)_{8'} \oplus (o)_8] + [(v)_{8'} \oplus (v)_8] , \\
(v)_{16} &= [(v)_{8'} \oplus (o)_8] + [(o)_{8'} \oplus (v)_8] , \\
(s)_{16} &= [(s)_{8'} \oplus (s)_8] + [(c)_{8'} \oplus (c)_8] , \\
(c)_{16} &= [(s)_{8'} \oplus (c)_8] + [(c)_{8'} \oplus (s)_8] .
\end{aligned} \tag{2.4}$$

The vectors of norm one in $\mathrm{SO}'(8)$ are the 4-vectors $\sqrt{2\alpha'}\mathbf{p}'_\mathbf{v}, \sqrt{2\alpha'}\mathbf{p}'_\mathbf{s}$ and $\sqrt{2\alpha'}\mathbf{p}'_\mathbf{c}$ as described above. We choose one vector $\sqrt{2\alpha'}\mathbf{p}'_\mathbf{v}$ and one vector $\sqrt{2\alpha'}\mathbf{p}'_\mathbf{s}$. One may equivalently choose $\sqrt{2\alpha'}\mathbf{p}'_\mathbf{c}$ instead of $\sqrt{2\alpha'}\mathbf{p}'_\mathbf{s}$.

Focusing on the first choice, we get the following truncation for the lattice partition functions

$$\begin{aligned}
o_{16} &\to v_8 , \qquad v_{16} \to o_8 , \\
s_{16} &\to -s_8 , \qquad c_{16} \to -c_8 .
\end{aligned} \tag{2.5}$$

It follows from the closure of the Lorentz algebra that states belonging to v_8 or o_8 are bosons while those belonging to the spinor partition functions s_8 and c_8 are space-time fermions. In accordance with the spin-statistic theorem we have flipped the sign in the partition function of the space-time spinor partition functions.[4]

The truncation eq. (2.5) does preserve the modular invariance of the original bosonic closed string theory. Under the S modular transformation $\tau \to -1/\tau$ the partition functions of the four sublattices $(o)_{16}$, $(v)_{16}$, $(s)_{16}$, $(c)_{16}$ transform as

$$
\begin{aligned}
o_{16} &\to \frac{1}{2}[o_{16} + v_{16} + s_{16} + c_{16}]\,, \\
v_{16} &\to \frac{1}{2}[o_{16} + v_{16} - s_{16} - c_{16}]\,, \\
s_{16} &\to \frac{1}{2}[o_{16} - v_{16} + s_{16} - c_{16}]\,, \\
c_{16} &\to \frac{1}{2}[o_{16} - v_{16} - s_{16} + c_{16}]\,,
\end{aligned}
\tag{2.6}
$$

as do the four SO(8) partition functions o_8, v_8, s_8 and c_8. From eq. (2.6) and the truncation-flip eq. (2.5), it is indeed easily proven [3, 8] that modular invariance is preserved under truncation, namely that the truncation-flip commutes with both $S(\tau \to -1/\tau)$ and $T(\tau \to \tau + 1)$.

The modular invariant compactifications with gauge symmetry $\mathcal{G}_L \times \mathcal{G}_R$ and $\mathcal{G}_R = E_8 \times \mathrm{SO}(16)$ yield all heterotic strings, supersymmetric or not, when the right sector is truncated according to eq. (2.5) [9, 3]. The other fermionic closed strings theories, namely type IIB, IIA, OB and OA, can be obtained from compactifications with $\mathcal{G} \times \mathcal{G}$ symmetry. Such compactifications are modular invariant for any semi-simple simply laced group $\mathcal{G}$ of rank d , if both $\sqrt{2\alpha'}\mathbf{p_L}$ and $\sqrt{2\alpha'}\mathbf{p_R}$ span the full weight lattice $\Lambda_{\mathrm{weight}}$ of $\mathcal{G}$, but are constrained to be in the same conjugacy class [10]. Such lattices will be referred to as EN lattices. Taking the EN lattice of $\mathcal{G} = E_8 \times \mathrm{SO}(16)$ and using eq. (2.5) in both sectors, one gets the non-supersymmetric OB theory. The same procedure applied to the EN lattice of $\mathcal{G} = E_8 \times E_8$ yields the supersymmetric IIB theory. The OA and IIA theories follow from the same bosonic theory by using eq. (2.5) in the right sector and interchanging s_8 with c_8 in the left one. This amounts to use $\sqrt{2\alpha'}\mathbf{p'_c}$ instead of $\sqrt{2\alpha'}\mathbf{p'_s}$ as the momentum kept in the left $\mathrm{SO}'(8)$ group. The subsequent analysis will however show that while the above asymmetric truncation for type IIA and OA is valid as long as only closed strings are considered, it cannot be used for analyzing their D-branes. We shall then see how to formulate a symmetric truncation for all cases. This will allow for a generalization of the D-brane analysis given in reference [3]. More importantly perhaps, it will reveal relations between fermionic theories encoded in their bosonic parents.

The four lattice partition functions o_{16}, v_{16}, s_{16}, c_{16} are the characters of level one representations of the affine Kac-Moody algebra $\widehat{\mathrm{so}}(16)$. They define an extended conformal algebra and all chiral operators describing interactions of the bosonic string in the right

[4] Note that this sign flip is consistent with the positive sign for the bosonic partition functions because of a sign ambiguity for the factors in the decomposition of the SO(16) lattice partition function in sum of products of $\mathrm{SO}'(8)$ and SO(8) partition functions.

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sector must obey the corresponding fusion rules. These are determined by the group multiplication table of the conjugacy classes of SO(16), that is of the centre $Z_2 \times Z_2$ of its universal covering. They are given by

$$
\begin{aligned}
[o_{16}][o_{16}] &= [v_{16}][v_{16}] = [s_{16}][s_{16}] = [c_{16}][c_{16}] = [o_{16}]\,, \\
[o_{16}][v_{16}] &= [v_{16}]\,, \qquad [o_{16}][s_{16}] = [s_{16}]\,, \qquad [o_{16}][c_{16}] = [c_{16}]\,, \\
[v_{16}][s_{16}] &= [c_{16}]\,, \qquad [v_{16}][c_{16}] = [s_{16}]\,, \qquad [s_{16}][c_{16}] = [v_{16}]\,.
\end{aligned} \tag{2.7}
$$

In this sector, the chiral operators describing the ten-dimensional fermionic string interactions can, in the light-cone gauge, be expressed in terms of bosonic operators acting on the fermionic subspace. These operators must therefore obey the fusion rules obtained by truncating the bosonic ones,

$$
\begin{aligned}
[v_8][v_8] &= [o_8][o_8] = [s_8][s_8] = [c_8][c_8] = [v_8]\,, \\
[v_8][o_8] &= [o_8]\,, \qquad [v_8][s_8] = [s_8]\,, \qquad [v_8][c_8] = [c_8]\,, \\
[o_8][s_8] &= [c_8]\,, \qquad [o_8][c_8] = [s_8]\,, \qquad [s_8][c_8] = [o_8]\,.
\end{aligned} \tag{2.8}
$$

In the fusion rules eq. (2.8) the rôle of the identity has been transferred from $[o_{16}]$ to the vector class $[v_8]$ [2, 11].

Similarly, for the left extended conformal algebra the fusion rules are

$$
\begin{aligned}
[\bar{o}_{16}][\bar{o}_{16}] &= [\bar{v}_{16}][\bar{v}_{16}] = [\bar{s}_{16}][\bar{s}_{16}] = [\bar{c}_{16}][\bar{c}_{16}] = [\bar{o}_{16}]\,, \\
[\bar{o}_{16}][\bar{v}_{16}] &= [\bar{v}_{16}]\,, \qquad [\bar{o}_{16}][\bar{s}_{16}] = [\bar{s}_{16}]\,, \qquad [\bar{o}_{16}][\bar{c}_{16}] = [\bar{c}_{16}]\,, \\
[\bar{v}_{16}][\bar{s}_{16}] &= [\bar{c}_{16}]\,, \qquad [\bar{v}_{16}][\bar{c}_{16}] = [\bar{s}_{16}]\,, \qquad [\bar{s}_{16}][\bar{c}_{16}] = [\bar{v}_{16}]\,.
\end{aligned} \tag{2.9}
$$

Under both the symmetric and asymmetric truncations, the fermionic fusion rules in the left sector are the antiholomorphic counterpart of the fusion rules eq. (2.8). Thus, as far as fusion rules are concerned symmetric and asymmetric truncations appear to be valid.

When D-branes are introduced the asymmetric truncation runs into problems. Indeed in that case, boundary conditions relate the left and right conformal families of the bosonic theories leaving only conformal families of a single algebra. The asymmetric truncations yielding type IIA and OA are then at best ambiguous and in fact turn out to be inconsistent for Dirichlet boundary conditions. To obtain all fermionic D-branes we must use the *symmetric* truncation and thus take eq. (2.5) in both sectors of the closed bosonic strings.

To realize this program we cannot restrict ourselves to EN lattices, and we must find new modular invariant compactifications on left and right $E_8 \times \mathrm{SO}(16)$ lattices. Here and in the rest of the paper, we only write down the SO(16) characters in the integrand of amplitudes: the 'dummy' E_8 character and the contribution of the eight non-compact dimensions are not displayed. There are *four* such modular invariant partition functions, given by

$$\mathrm{OB}_b = \bar{o}_{16}\; o_{16} + \bar{v}_{16}\; v_{16} + \bar{s}_{16}\; s_{16} + \bar{c}_{16}\; c_{16}\,, \tag{2.10}$$

$$\mathrm{OA}_b = \bar{o}_{16}\; o_{16} + \bar{v}_{16}\; v_{16} + \bar{s}_{16}\; c_{16} + \bar{c}_{16}\; s_{16}\,, \tag{2.11}$$

$$\mathrm{IIB}_b = \bar{o}_{16}\; o_{16} + \bar{s}_{16}\; o_{16} + \bar{o}_{16}\; s_{16} + \bar{s}_{16}\; s_{16}\,, \tag{2.12}$$

$$\mathrm{IIA}_b = \bar{o}_{16}\; o_{16} + \bar{c}_{16}\; o_{16} + \bar{o}_{16}\; s_{16} + \bar{c}_{16}\; s_{16}\,. \tag{2.13}$$

The OB_b and the IIB_b theories describe the previously mentioned compactifications on the EN lattice of SO(16) and on its E_8 sublattice. One can show by inspection that the two other theories, namely OA_b and IIA_b, correspond to even self-dual lorentzian lattices and are thus modular invariant too. This property also follows from the existence of conformal σ-model actions. Namely the four compactifications eqs.(2.10)–(2.13) may be expressed in terms of actions [12] (from now on we choose $\alpha' = 1/2$)

$$S = \frac{-1}{2\pi} \int d\sigma d\tau \Big[\Big\{ g_{ab} \partial_\alpha X^a \partial^\alpha X^b + b_{ab} \epsilon^{\alpha\beta} \partial_\alpha X^a \partial_\beta X^b \Big\} + \\ + \eta_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu \Big] , \tag{2.14}$$

with g_{ab} a constant metric and b_{ab} a constant antisymmetric tensor in the compact directions ($a, b = 1, \ldots, 16$), $\eta_{\mu\nu} = (-1; +1, \ldots)$ for $\mu, \nu = 1, \ldots, 10$ and $0 \le \sigma \le \pi$. The fields X^a are periodic with period 2π. In this formalism the left and right momenta are given by

$$\begin{aligned} \mathbf{p_R} &= \left[\frac{1}{2} m_b + n^a (b_{ab} + g_{ab}) \right] \mathbf{e^b} , \\ \mathbf{p_L} &= \left[\frac{1}{2} m_b + n^a (b_{ab} - g_{ab}) \right] \mathbf{e^b} , \end{aligned} \tag{2.15}$$

where $\{\mathbf{e^a}\}$ is the lattice-dual basis of the basis $\{\mathbf{e_a}\}$ defining the configuration space torus

$$\mathbf{x} \equiv \mathbf{x} + 2\pi n^a \mathbf{e_a} \qquad n^a \in \mathcal{Z} , \tag{2.16}$$

and the lattice metric is given by

$$g_{ab} = \mathbf{e_a} . \mathbf{e_b} . \tag{2.17}$$

Explicit forms of the g_{ab} and b_{ab} tensors for the four models will be given later.

Using the *same* universal truncation eq. (2.5), we now truncate these theories in both left and right sectors . We get

$$\mathrm{OB}_b \to \bar{o}_8 \; o_8 + \bar{v}_8 \; v_8 + \bar{s}_8 \; s_8 + \bar{c}_8 \; c_8 \equiv OB , \tag{2.18}$$

$$\mathrm{OA}_b \to \bar{o}_8 \; o_8 + \bar{v}_8 \; v_8 + \bar{s}_8 \; c_8 + \bar{c}_8 \; s_8 \equiv \mathrm{OA} , \tag{2.19}$$

$$\mathrm{IIB}_b \to \bar{v}_8 \; v_8 - \bar{s}_8 \; v_8 - \bar{v}_8 \; s_8 + \bar{s}_8 \; s_8 \equiv \mathrm{IIB} , \tag{2.20}$$

$$\mathrm{IIA}_b \to \bar{v}_8 \; v_8 - \bar{c}_8 \; v_8 - \bar{v}_8 \; s_8 + \bar{c}_8 \; s_8 \equiv \mathrm{IIA} . \tag{2.21}$$

Starting from the four possible bosonic string theories compactified on left and right $E_8 \times \mathrm{SO}(16)$ lattices we thus obtain the four consistent non-heterotic 10-dimensional fermionic theories. The compactification eq. (2.10) on the $\mathcal{G} = E_8 \times \mathrm{SO}(16)$ EN lattice, which defines the OB_b theory, yields the same gauge group as the OA_b theory defined by the compactification eq. (2.11), namely $\mathcal{G} \times \mathcal{G}$ with $\mathcal{G} = E_8 \times \mathrm{SO}(16)$. Similarly the compactification eq. (2.12) on the $\mathcal{G} = E_8 \times E_8$ EN lattice has the same gauge group as the IIA_b theory defined by the compactification eq. (2.13), namely $\mathcal{G} \times \mathcal{G}$ with $\mathcal{G} = E_8 \times E_8$. Comparaison of eqs. (2.18) and (2.19) with eqs. (2.20) and (2.21) shows that the enhanced gauge symmetry in the bosonic parents from SO(16) to E_8 signals the onset of supersymmetry for their fermionic offsprings. This correspondence between supersymmetry in fermionic strings and the occurrence of an E_8 gauge symmetry for their bosonic parents (modulo the 'dummy' E_8) holds for all closed string theories, including the heterotic ones [3]. In the next sections it will be seen that the 'breaking' of supersymmetry is *always*, both in closed and open string sectors, translated to the breaking of the E_8 symmetry to its subgroup SO(16) in the bosonic parent theories. This feature is indicative of the fundamental rôle played in the bosonic theory by the SO(16) weight lattice, of which the E_8 lattice is a sublattice, in linking together their fermionic subspaces.

3. D9-branes and torus geometry

In this section and in the following ones we discuss the bosonic D-branes of the four different bosonic theories compactified on $E_8 \times \mathrm{SO}(16)$ lattices. D-branes break the $\mathcal{G} \times \mathcal{G}$ symmetry of the closed strings to $\mathcal{G}$ and may even break $\mathcal{G}$ itself. We shall indeed find that, when the unbroken $\mathcal{G}$ is E_8 (modulo the 'dummy' E_8), breaking of E_8 to SO(16) may occur leading to a concomitant breaking of supersymmetry in the fermionic sector. Such breaking will manifest itself by the appearance of the partition function v_{16} in the open string sector of the IIB_b (or the IIA_b) theory (see table 4 below), and hence, from the truncation $v_{16} \to o_8$, by a tachyon in the IIB (or the IIA) theory.

In this section, we relate the properties of the bosonic D9-branes to the geometry of the configuration space torus characterizing each compactification. These tori are linked to each other through global properties of the universal covering group $\widetilde{\mathrm{SO}}(16)$. We then perform the symmetric truncation and obtain the space-filling D9-branes of the different fermionic theories.

We want to find the bosonic D9-branes with Dirichlet boundary conditions in the 16-dimensional compact space. The amplitudes $\mathcal{A}_{\mathrm{tree}}$ describing the D9-branes in the tree channel are obtained from the torus partition functions eqs. (2.10)–(2.13) by imposing Dirichlet boundary conditions on the compact space. For open strings the latter do not depend on b_{ab} and are given by

$$\partial_\tau X^a = 0\,, \tag{3.1}$$

where τ is the worldsheet time coordinate and σ the space one. Using the worldsheet duality which interchanges the rôles of τ and σ, these equations yield the following relation between the left and right momenta:

$$\mathbf{p_L} - \mathbf{p_R} = 0\,, \tag{3.2}$$

as well as a match between left and right oscillators in the tree channel. The conditions eq. (3.2) determine the closed strings which propagate in the annulus amplitude. Imposing them on the four tori amounts to keep *all* characters which appear diagonally in eqs. (2.10)–(2.13). Up to a normalization α, the annulus amplitudes, written as closed string tree

	$2^5\mathcal{A}_{\rm tree}$	$\mathcal{A}$
OB_b	$(1/2)\,(o_{16}+v_{16}+s_{16}+c_{16})$	o_{16}
OA_b	$o_{16}+v_{16}$	$o_{16}+v_{16}$
IIB_b	$o_{16}+s_{16}$	$o_{16}+s_{16}$
IIA_b	$2\,o_{16}$	$o_{16}+v_{16}+s_{16}+c_{16}$

Table 1: Bosonic D9-brane amplitudes.

amplitudes, are

$$\begin{aligned}
\mathcal{A}_{\rm tree}(\mathrm{OB}_b) &= \alpha_{\mathrm{OB}_b}(o_{16}+v_{16}+s_{16}+c_{16})\,,\\
\mathcal{A}_{\rm tree}(\mathrm{OA}_b) &= \alpha_{\mathrm{OA}_b}(o_{16}+v_{16})\,,\\
\mathcal{A}_{\rm tree}(\mathrm{IIB}_b) &= \alpha_{\mathrm{IIB}_b}(o_{16}+s_{16})\,,\\
\mathcal{A}_{\rm tree}(\mathrm{IIA}_b) &= \alpha_{\mathrm{IIA}_b}o_{16}\,.
\end{aligned}\tag{3.3}$$

The normalization is determined by rewriting the amplitudes eq. (3.3) as loop amplitudes of open strings and by expressing that these partition functions count the states of a single string (i.e. without Chan-Paton multiplicity). To express $\mathcal{A}_{\rm tree}$ as a loop amplitude $\mathcal{A}$, one performs a change of variable and the S-transformation on the modular parameter ($\tau\to-1/\tau$). We get[5]

$$\begin{aligned}
\mathcal{A}(\mathrm{OB}_b) &= 2^5\,\alpha_{\mathrm{OB}_b}(2\,o_{16})\,,\\
\mathcal{A}(\mathrm{OA}_b) &= 2^5\,\alpha_{\mathrm{OA}_b}(o_{16}+v_{16})\,,\\
\mathcal{A}(\mathrm{IIB}_b) &= 2^5\,\alpha_{\mathrm{IIB}_b}(o_{16}+s_{16})\,,\\
\mathcal{A}(\mathrm{IIA}_b) &= 2^5\,\alpha_{\mathrm{IIA}_b}\left(\frac{1}{2}\right)(o_{16}+v_{16}+s_{16}+c_{16})\,.
\end{aligned}\tag{3.4}$$

To describe, in each theory, *one* elementary D9-brane, we see that we have to choose $\alpha_{\mathrm{OB}_b}=2^{-6}, \alpha_{\mathrm{OA}_b}=\alpha_{\mathrm{IIB}_b}=2^{-5}$ and $\alpha_{\mathrm{IIA}_b}=2^{-4}$. Table I gives the amplitudes of the corresponding elementary D9-branes.

We now use these results to determine the configuration space torus on which each of the four bosonic theories eqs. (2.10)–(2.13) is defined. The tori define lattices with basis vectors $\{2\pi\mathbf{e_a}\}$ according to eq. (2.16). In order to find the lattice corresponding to each theory, we first note that the Dirichlet condition eq. (3.2) reduces eq. (2.15) to $\mathbf{p_L}=\mathbf{p_R}=(1/2)m_a\mathbf{e^a}$ (independent of b_{ab}). Using the general expression for lattice partition functions eq. (2.2), we read off for each model the dual of its SO(16) weight sublattice from the four tree amplitudes in table 1. We then deduce the $\{\mathbf{e_a}\}$ from the duality between the root lattice $(o)_{16}$ and the weight lattice $(o)_{16}+(v)_{16}+(c)_{16}+(s)_{16}$, and from the self-duality of $(o)_{16}+(v)_{16}$ and $(o)_{16}+(s)_{16}$. We get

$$\mathbf{e_a}=\left(\frac{1}{2}\right)\mathbf{w_a}\,,\tag{3.5}$$

[5]The factor 2^5 comes from the change of variable.

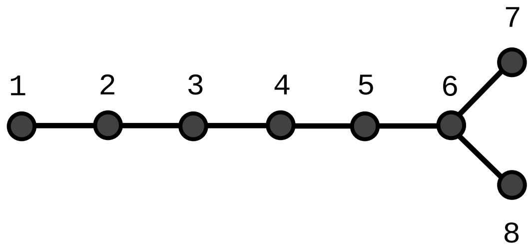

Figure 1: Dynkin diagram of SO(16).

where the $\mathbf{w_a}$ are weight vectors forming a basis of a sublattice $(r)_{16}$ of the weight lattice of SO(16). The sublattice $(r)_{16}$ for each theory is

$$\begin{aligned}
(\mathrm{OB}_b) &: (r)_{16} = (o)_{16}\,, \\
(\mathrm{OA}_b) &: (r)_{16} = (o)_{16} + (v)_{16}\,, \\
(\mathrm{IIB}_b) &: (r)_{16} = (o)_{16} + (s)_{16}\,, \\
(\mathrm{IIA}_b) &: (r)_{16} = (o)_{16} + (v)_{16} + (s)_{16} + (c)_{16}\,.
\end{aligned} \tag{3.6}$$

Alternatively we may derive these results from the loop channel and read off the $\mathbf{e_a}$ directly from table 1. Indeed, for Dirichlet boundary conditions, the hamiltonian eigenvalues for open strings are independent of b_{ab} and, for zero-modes, are equal to $(2\mathbf{l}.2\mathbf{l})/2$ where $\mathbf{l} = n^a\mathbf{e_a}$ is the winding lattice. Comparing the amplitudes $\mathcal{A}$ in table 1 to the general expression for lattice partition functions eq. (2.2), we see that $2\mathbf{e_a} = \mathbf{w_a}$, in accordance with eq. (3.5).

Many properties of these tori can be visualized by a suitable projection.

Consider the SO(16) Dynkin diagram in figure 1. We project the SO(16) weight lattice on the plane of the simple roots $\mathbf{r}_7 = (0,0,0,0,0,0,1,-1)$ and $\mathbf{r}_8 = (0,0,0,0,0,0,1,1)$, along the simple roots $\mathbf{r}_1,\ldots,\mathbf{r}_6$. This projection maps the SO(16) onto the SO(4) weight lattice: ($o_{16} \to o_4$, $v_{16} \to v_4$, $s_{16} \to c_4$, $c_{16} \to s_4$). This is easily verified by computing the projections $\mathbf{w}^i_p$ of the fundamental weights $\mathbf{w}^i$ $(i = 1,\ldots,7,8)$: $\mathbf{w}^i_p = (\mathbf{w}^i.\mathbf{w}^7)\,\mathbf{r}_7 + (\mathbf{w}^i.\mathbf{w}^8)\,\mathbf{r}_8$.

The projection preserves the root length and the number of weight lattice points in each unit cell of the projected sublattices $(r)_{16}$. Hence it gives the correct volumes of these unit cells (although the lengths of spinor weights are not preserved). The projected SO(16) weight lattice is depicted in figure 2.

The figure shows that the unit SO(16) cell volumes ξ_i of the four bosonic theories are given by

$$\xi_{\mathrm{OB}_b} = 2\,, \qquad \xi_{\mathrm{OA}_b} = 1\,, \qquad \xi_{\mathrm{IIB}_b} = 1\,, \qquad \xi_{\mathrm{IIA}_b} = \frac{1}{2}\,. \tag{3.7}$$

Restoring the α' dependence and taking into account the contribution of the E_8 lattice, one gets the volumes V_i of the configuration space tori

$$V_i = (2\pi\alpha'^{\,1/2})^{16}2^{-8}\xi_i\,. \tag{3.8}$$

These volumes are thus related by

$$V_{\mathrm{OB}_b} = 2V_{\mathrm{OA}_b} = 2V_{\mathrm{IIB}_b} = 4V_{\mathrm{IIA}_b}\,. \tag{3.9}$$

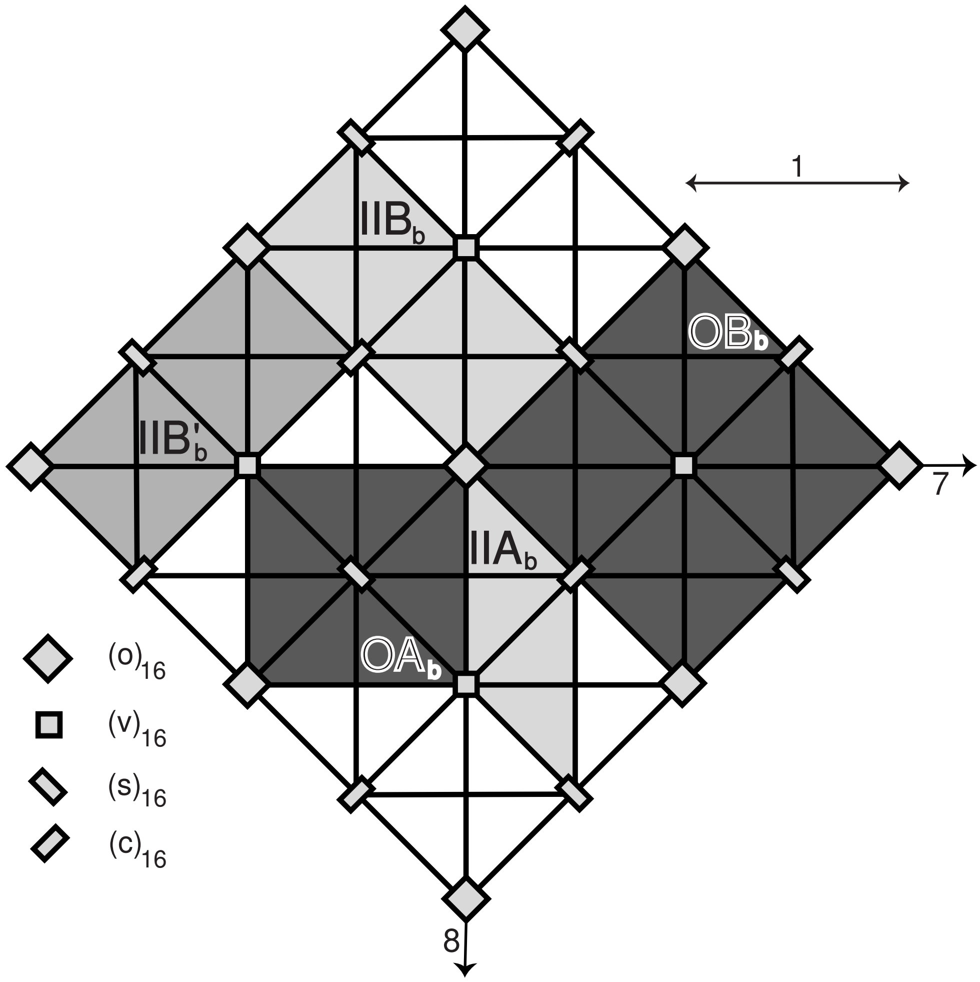

Figure 2: Projected weight lattice of SO(16) in the $r_7 - r_8$ plane. We see from eqs. (2.16), (3.5) that the volumes ξ_i of the unit cells, exhibited in shaded areas, must be multiplied by $(2\pi)^8.2^{-8}$ to yield the SO(16) compactification space torus volume of the four bosonic theories (in units where $\alpha' = 1/2$). The theories IIB and IIB' are isomorphic and differ by the interchange of s_{16} and c_{16}.

The tori $\tilde{t}$ of the four bosonic theories are, as group spaces, the maximal toroids $\widetilde{\mathcal{T}}/Z_c$ of the locally isomorphic groups $E_8 \times \widetilde{\mathrm{SO}}(16)/Z_c$ where Z_c is a subgroup of the centre $Z_2 \times Z_2$ of the universal covering group $\widetilde{\mathrm{SO}}(16)$. We write

$$\begin{aligned}
\tilde{t}\,(\mathrm{OB}_b) &= \widetilde{\mathcal{T}}\,, \\
\tilde{t}\,(\mathrm{OA}_b) &= \frac{\widetilde{\mathcal{T}}}{Z_2^d}\,, \\
\tilde{t}\,(\mathrm{IIB}_b) &= \frac{\widetilde{\mathcal{T}}}{Z_2^+} \qquad \text{or} \qquad \frac{\widetilde{\mathcal{T}}}{Z_2^-}\,, \\
\tilde{t}\,(\mathrm{IIA}_b) &= \frac{\widetilde{\mathcal{T}}}{(Z_2 \times Z_2)}\,,
\end{aligned} \tag{3.10}$$

where $Z_2^d = \mathrm{diag}(Z_2 \times Z_2)$ and the superscripts $\pm$ label the two isomorphic IIB$_b$ theories

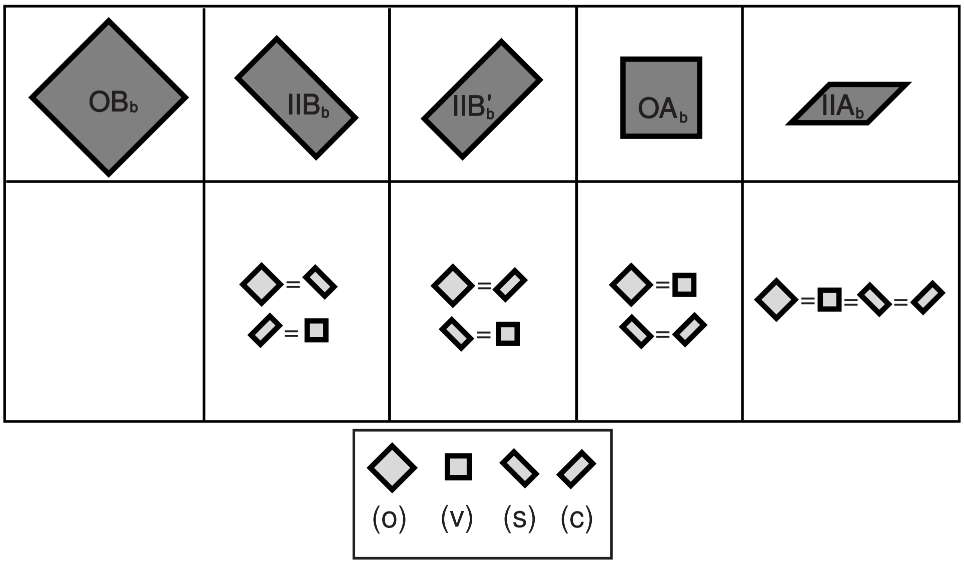

Figure 3: Identifications of centre elements of $\widetilde{\mathrm{SO}}(16)$ engendering the bosonic theories IIB$_b$, (IIB$'_b$), OA$_b$ and IIA$_b$ from the mother theory OB$_b$. The dark shaded figure s represent the tori of these bosonic theories in the SO(16) compact space.

obtained by interchanging $(s)_{16}$ and $(c)_{16}$.

There is thus a unified picture for the four theories related to the global properties of the SO(16) group. The OB$_b$ theory built upon $\widetilde{\mathcal{T}}$ plays in some sense the rôle of the 'mother theory' of the others. One may view the different maximal toroids eq. (3.10) as resulting from the identification of centre elements of $\widetilde{\mathrm{SO}}(16)$, which are represented by weight lattice points, with its unit element. These identifications give rise to the smaller shaded cells of figure 2. In this way, the unit cell of the IIB$_b$ theory is obtained from the OB$_b$ one by identifying the (o) and (s) lattice points[6] (or alternatively the (o) and the (c) lattice points) and therefore also the (v) and (c) lattice (or the (v) and the (s) lattice), as seen in figure 2. It is therefore equal to the unit cell of the E_8 lattice.[7] The unit cell of the OA$_b$ theory is obtained by identification of (o) and (v), and of (s) and (c).

Finally, the unit cell of the IIA$_b$ theory results from the identification of (o), (v), (s) and (c) and no non-trivial centre element is left. These results are summarized in figure 3.

We now evaluate the tension of the D9-branes. Tree amplitudes $\mathcal{A}_{\text{tree}}$ are proportional to the square of the D-brane tension T. We can immediately deduce from table 1 the following relations between the tensions of the elementary D9-branes of the different theories

$$\sqrt{2}\,T_{\mathrm{OB}_b} = T_{\mathrm{OA}_b} = T_{\mathrm{IIB}_b} = \left(\frac{1}{\sqrt{2}}\right) T_{\mathrm{IIA}_b}\,. \tag{3.11}$$

To get their values, we recall that the tension T^{bosonic}_{Dp} of a Dp-brane in the 26-dimensional

[6]From now on we drop the subscript 16 in the labeling of the lattice $(i)_{16}$.
[7]The latter however does not contain (v) and (c) lattice points.

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uncompactified theory is [13]

$$T_{Dp}^{\text{bosonic}} = \frac{\sqrt{\pi}}{2^4 \kappa_{26}} \left(2\pi\alpha'^{1/2}\right)^{11-p}, \tag{3.12}$$

where $\kappa_{26}^2 = 8\pi G_{26}$ and G_{26} is the newtonian constant in 26 dimensions. The tensions of the Dirichlet D9-branes of the four compactified theories are obtained from eq. (3.12) by expressing κ_{26} in term of the 10-dimensional coupling constant κ_{10}. Recalling that $\kappa_{26} = \sqrt{V}\kappa_{10}$ where V is the volume of the configuration space torus, one finds, using eqs. (3.7) and (3.8)

$$T_{\mathrm{OB}_b} = \frac{\sqrt{\pi}}{\sqrt{2}\kappa_{10}} \left(2\pi\alpha'^{1/2}\right)^{-6}, \tag{3.13}$$

$$T_{\mathrm{OA}_b} = T_{\mathrm{IIB}_b} = \frac{\sqrt{\pi}}{\kappa_{10}} \left(2\pi\alpha'^{1/2}\right)^{-6}, \tag{3.14}$$

$$T_{\mathrm{IIA}_b} = \frac{\sqrt{2}\sqrt{\pi}}{\kappa_{10}} \left(2\pi\alpha'^{1/2}\right)^{-6}. \tag{3.15}$$

These are consistent with eq. (3.11).

We now perform the truncation eq. (2.5) on the loop amplitudes $\mathcal{A}$ listed in table 1. This yields the open string spectrum of the fermionic D9-branes

$$\mathcal{A}(\mathrm{OB}_b) \to v_8 \equiv \mathcal{A}(\mathrm{OB}), \tag{3.16}$$

$$\mathcal{A}(\mathrm{OA}_b) \to o_8 + v_8 \equiv \mathcal{A}(\mathrm{OA}), \tag{3.17}$$

$$\mathcal{A}(\mathrm{IIB}_b) \to v_8 - s_8 \equiv \mathcal{A}(\mathrm{IIB}), \tag{3.18}$$

$$\mathcal{A}(\mathrm{IIA}_b) \to o_8 + v_8 - s_8 - c_8 \equiv \mathcal{A}(\mathrm{IIA}). \tag{3.19}$$

These are indeed the amplitudes describing the fermionic D9-branes of respectively OB, OA, IIB and IIA.

Furthermore tension is conserved in the truncation as proven in reference [3]. The tensions of the different bosonic D9-branes given in eqs. (3.13)–(3.15) are thus equal, when measured with the same gravitational constant κ_{10}, to the tensions of the corresponding fermionic D9-branes [14, 15]. This is indeed a correct prediction.

Up to now, we studied the properties of a single 'elementary' D9-brane for the four SO(16) bosonic strings and for the corresponding fermionic theories. We now generalize the analysis to encompass several D-branes.

First, we remark that the relative position of the different D9-branes in the eight compact dimensions of the SO(16) torus, depicted in figure 4, is not arbitrary. Group symmetry requires that the partition function of an open string with end points on D9-branes be a linear combination of the four SO(16) characters. The vector $\mathbf{d}$ separating the two points where two distinct D9-branes meet the SO(16) torus determines the partition function of the string starting at one point and ending at the other after winding any number of times around the torus. The smallest eigenvalue of the string hamiltonian is $(1/2)\,\mathbf{d}.\mathbf{d}/\pi^2$. Therefore $\mathbf{d}/\pi$ must be a weight and the D9-branes can only be separated in the compact space (rescaled, as in figure 2, by a factor π^{-8}) by a weight vector. Consider

Figure 4: The figure depicts the localization of the (o), (v), (s), (c) D-branes in the compact SO(16) torus. The cylinder represents the 9-dimensional non-compact space and the circle the 8-dimensional compact SO(16) torus. 'Charge conjugate' branes are connected by arrowed lines.

for instance two branes in the OB_b theory, one located at (o) and the other located at (v). The partition function of a string beginning and ending on the same brane is o_{16}, while the partition function of an open string stretching between them is v_{16}. For the other theories, the partition function of a string beginning and ending on the same brane will then contain, in addition to o_{16}, the characters corresponding to the strings stretched between (o) and all points identified with (o). This can be checked by comparing the identifications indicated in figure 3 with the partition functions $\mathcal{A}$ listed in table 1.

If one chooses the location of one elementary brane as the origin of the weight lattice, the other D9-branes can then only meet the SO(16) torus (rescaled by π^{-8}) at a weight lattice point. The number of distinct elementary branes is, for each of the four bosonic theories, equal to the number of distinct weight lattice points in the unit cell. For the mother theory OB_b there are four possible elementary D9-branes. We label them by their positions in the unit cell, namely by (o), (v), (s) and (c). Note that these weight lattice points represent the centre elements of the $\widetilde{\mathrm{SO}}(16)$. For the other theories the unit cells are smaller and there are fewer possibilities. The unit cell of the IIB_b theory allows only for two distinct branes $(o) = (s)$, $(c) = (v)$, as seen from figure 3. For IIB'_b, one simply interchanges (s) and (c). Similarly for the OA_b theory, we have the two branes $(o) = (v)$ and $(s) = (c)$, and finally for the 'smallest' theory IIA_b, we have only one elementary brane $(o) = (v) = (s) = (c)$.

It is interesting to note that the charge conjugation of the truncated fermionic strings is encoded in their bosonic parents. A brane sitting at (v) can always be joined by an open string to a brane sitting at (o). The partition function of such a string is given by the character v_{16} and therefore the two branes can exchange closed strings with tree amplitude $\mathcal{A}_{\mathrm{tree}} = o_{16} + v_{16} - s_{16} - c_{16}$ as seen from the S-transformation eq. (2.6). Namely the closed string exchange describing the interaction between these two branes has opposite sign for the (s) and (c) contribution as compared to the closed string exchange between D9-branes located at the same point. Using eq. (2.5), we see that this shift of sign persists in the fermionic theories where the above tree amplitude becomes $o_8 + v_8 + s_8 + c_8$. This shift of sign is the RR-charge conjugation between fermionic D9-branes. It is encoded in the

	$\mathcal{A}^{\rm trunc}$	Fermionic D9-branes	stability
$\mathrm{OB}_b \to \mathrm{OB}$	v_8	$D_1^+ + D_2^+ + D_1^- + D_1^-$	stable
$\mathrm{OA}_b \to \mathrm{OA}$	$v_8 + o_8$	$D_1^0 + D_2^0$	unstable
$\mathrm{IIB}_b \to \mathrm{IIB}$	$v_8 - s_8$	$D^+ + D^-$	stable
$\mathrm{IIA}_b \to \mathrm{IIA}$	$v_8 + o_8 - s_8 - c_8$	D^0	unstable

Table 2: Fermionic D9-brane amplitudes.

bosonic string as a shift by the lattice vector (v) (see figure 4). In particular, when (o) and (v) are identified, all branes of the fermionic offsprings are neutral. These are always unstable branes, as the truncation of v_{16} is o_8 and contains a tachyon. Charged branes are always stable.

The distinct fermionic D9-branes and their charge conjugation properties can thus be directly read off from figure 2. They are summarized in table 2, where the charge is indicated by a superscript $+, -$ or 0, and additional quantum numbers by a subscript.

Having discussed the elementary space-filling D9-branes of the different bosonic theories and their fermionic offsprings we now turn to the discussion of Chan-Paton multiplicities and the general annulus amplitudes of each bosonic theory.

The general direct annulus amplitude of the mother OB_b theory can be written as

$$\mathcal{A} = \sum_{l,m,i} \mathcal{A}^i{}_{lm}\, n^l \bar{n}^m \, i_{16} \,, \tag{3.20}$$

where the $n^l, \bar{n}^m$ are the Chan-Paton multiplicities (one for each end of the open strings) and the $\mathcal{A}^i{}_{lm}$ are coefficients. The sum is over the lattice points $(l), (m)$ in the unit cell and over the characters i_{16}; $l, m, i = o, v, s, c$. As seen from figure 2, a string stretched between branes located at lattice points $(l), (m)$ has end points separated by a weight vector equal to a vector joining (o) to some lattice point (i). Its partition function is i_{16}. Hence, the coefficients $\mathcal{A}^i{}_{lm}$ are given by the multiplication table of the conjugacy classes or, equivalently, by the fusion rules eq. (2.7). Writing these fusion rules as

$$[i][j] = \sum_k \mathcal{N}_{ij}^{\ \ k}\, [k] \,, \tag{3.21}$$

where the $\mathcal{N}^{ij}_{\ \ k}$ are the integer fusion-rule coefficients we have

$$\mathcal{A}^k_{\ ij} = \mathcal{N}_{ij}^{\ \ k} \,. \tag{3.22}$$

We recover, in this particular case, the Cardy solution for the annulus for diagonal conformal field theories [16, 17].

Explicitly, the general OB_b amplitude eq. (3.20) takes the form

$$\begin{aligned} \mathcal{A}(\mathrm{OB}_b) = {} & (n_o\bar{n}_o + n_v\bar{n}_v + n_s\bar{n}_s + n_c\bar{n}_c)\, o_{16} + \\ & + (n_o\bar{n}_v + n_v\bar{n}_o + n_s\bar{n}_c + n_c\bar{n}_s)\, v_{16} + \\ & + (n_o\bar{n}_s + n_s\bar{n}_o + n_v\bar{n}_c + n_c\bar{n}_v)\, s_{16} + \\ & + (n_o\bar{n}_c + n_c\bar{n}_o + n_v\bar{n}_s + n_s\bar{n}_v)\, c_{16} \,. \end{aligned} \tag{3.23}$$

Performing the identifications depicted in figure 3, we immediately get from eq. (3.23) the amplitudes for the other theories

$$\begin{aligned}\mathcal{A}(\mathrm{OA}_b) &= (n_o\bar{n}_o + n_s\bar{n}_s)\,(o_{16}+v_{16}) + (n_o\bar{n}_s + n_s\bar{n}_o)\,(s_{16}+c_{16})\,, & (3.24)\\ \mathcal{A}(\mathrm{IIB}_b) &= (n_o\bar{n}_o + n_v\bar{n}_v)\,(o_{16}+s_{16}) + (n_o\bar{n}_v + n_v\bar{n}_o)\,(v_{16}+c_{16})\,, & (3.25)\\ \mathcal{A}(\mathrm{IIA}_b) &= n_o\bar{n}_o\,(o_{16}+v_{16}+s_{16}+c_{16})\,. & (3.26)\end{aligned}$$

Note that in the identification, care has been exercised not to overcount the branes. Namely in going from eq. (3.23) to eq. (3.24) by identifying (v), (c) to (o), (s), one must put n_v and n_c equal to zero as these branes have already been counted in n_o and n_s. Similarly, one put n_s and n_c to zero in eq. (3.25) and n_v, n_s and n_c to zero in eq. (3.26).

These results may be viewed as particular realizations of boundary conformal field theory. eqs. (3.23)–(3.26) have the structure generalizing eq. (3.20)

$$\mathcal{A} = \sum_{a,b,i} \mathcal{A}^i{}_{ab} n^a \bar{n}^b\, i_{16}\,. \tag{3.27}$$

The indices a and b are 'charge' indices representing the branes where the open string may end, i labels the characters and the $\mathcal{A}^i{}_{ab}$ are integer-valued coefficients [4]. In the OB_b case, as discussed above, (see also figure 2) there are four possible locations for the open string end points and the charge indices a and b take the same four possible values o, v, s and c as i. For the three other models we have less charge indices than character labels. Following the discussion on the possible location of the D9-branes above, the OA_b theory and the IIB_b have both two charge indices, respectively $a = o, s$ and $a = o, v$. The IIA_b theory has only one possible charge label $a = o$.

Finally, the general amplitudes of the fermionic offsprings are obtained from eqs. (3.23)–(3.26) by the truncation eq. (2.5).

4. The even D-brane paradox

In this section we discuss the truncation of bosonic D-branes to lower dimensional fermionic Dp-branes (p<9). This is a non-trivial problem for the following reason. In fermionic string theories, a T-duality interchanges type IIA with IIB, and type OA with OB while transmuting D9-branes to D8-branes without changing their corresponding $\mathcal{A}^{\mathrm{trunc}}$ amplitudes[8] given in table II, although the latter may be conveniently rewritten in terms of representations of lower dimensional orthogonal groups [18]. This interchange is generally viewed as the compatibility of T-duality with world-sheet supersymmetry. It can also be understood in the light-cone gauge, or equivalently in the truncated theory, as the consistency of T-duality with the closure of the algebra $\mathrm{so}_{\mathrm{diag}}(8) = \mathrm{diag}[\mathrm{so}_{\mathrm{trans}}(8) \times \mathrm{so}_{\mathrm{int}}(8)]$ extended to the Lorentz algebra $\mathrm{so}(9,1)$ (see appendix A).

Therefore, *in any given fermionic theory*, the amplitudes of Dp-branes *whose p have a definite parity* are essentially the same (see footnote 8), but differ from the amplitudes of Dp-branes of the other parity.

[8] The only change in the integrand of the amplitudes is the power law of the moduli which, along with the Dedekind functions, are omitted throughout the paper.

	$\mathcal{A}_9 = \mathcal{A}_8$	$\mathcal{A}_8^{\text{trunc}}$		$\mathcal{A}_8'^{\,trunc}$
OB_b	o_{16}	v_8	OB	$o_8 + v_8$
OA_b	$o_{16} + v_{16}$	$o_8 + v_8$	OA	v_8
IIB_b	$o_{16} + s_{16}$	$v_8 - s_8$	IIB	$o_8 + v_8 - s_8 - c_8$
IIA_b	$o_{16} + v_{16} + s_{16} + c_{16}$	$o_8 + v_8 - s_8 - c_8$	IIA	$v_8 - s_8$

Table 3: The D8-brane puzzle.

	$\mathcal{A}_p\,,\quad p$ odd	$\mathcal{A}'_p\,,\quad p$ even
OB_b	o_{16}	$o_{16} + v_{16}$
OA_b	$o_{16} + v_{16}$	o_{16}
IIB_b	$o_{16} + s_{16}$	$o_{16} + v_{16} + s_{16} + c_{16}$
IIA_b	$o_{16} + v_{16} + s_{16} + c_{16}$	$o_{16} + s_{16}$

Table 4: Even and odd dimensional bosonic parent brane amplitudes.

This is in contrast with parent bosonic theories, where in a given theory $\mathcal{A}_p = \mathcal{A}_{p-1}$ for $p \leq 9$. Bosonic D8-branes are thus, as the D9 branes, uniquely determined by imposing Dirichlet boundary conditions in the 16-dimensional compact space. So, if we were to obtain a fermionic D8-brane[9] from the truncation $\mathcal{A}_8^{\text{trunc}}$ of a D8-brane in the bosonic parent theory, we would get a wrong result. The loop amplitude of a fermionic D8-brane, $\mathcal{A}_8'^{\,trunc}$ is different from $\mathcal{A}_8^{\text{trunc}}$, as shown in table 3.

The puzzle is twofold. On the one hand, we appear to lack, in a *given bosonic parent theory*, a whole family of branes with $p+1$ Neumann boundary conditions in the ten non-compact space-time dimensions whose lattice partition function coincides with the D9 lattice partition function of a *different* parent bosonic theory, namely the one obtained by interchanging $\text{OB}_b \Leftrightarrow \text{OA}_b$ and $\text{IIB}_b \Leftrightarrow \text{IIA}_b$! On the other hand, we get irrelevant D-branes for p even.

The fate of the irrelevant branes is easily understood from the fact that T-duality in the bosonic string does not change the bosonic parent theory. As $\mathcal{A}_8$ can be obtained from $\mathcal{A}_9$ from such T-duality, it is clear that a truncation leading to $\mathcal{A}_8^{\text{trunc}}$ in table 3 would be inconsistent with the Lorentz invariance of the truncated theory. Hence D8-branes, and more generally Dp-branes with p even disappear from the fermionic subspace.

The 'missing' loop amplitudes $\mathcal{A}'_p$ are shown in table 4. We shall show in the next section that the p-even amplitudes $\mathcal{A}'_p$ are obtained from bosonic D(p+8)-branes, namely from branes which wrap once around the SO(16) compact dimensions. On the other hand, the bosonic origin of the p-odd amplitudes is clear. They arise from the truncation of bosonic Dp-branes with the same lattice partition function as the bosonic D9-branes. It will be seen below that the correspondence between the p-odd and the p-even amplitudes, exhibited in table 4, is rooted in the existence of 'odd' E-dualities which interchange simultaneously the OB_b and OA_b (or IIB_b and IIA_b) theories, and the Dirichlet branes with the wrapped ones in the SO(16) compact dimensions.

[9]The argument is easily generalized to any even-dimensional brane.

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This solves the above puzzle. Completion of the T-duality in ten dimensions by an 'odd' E-duality in the eight dimensions compactified on an SO(16) torus yields all the bosonic branes listed in table 4 from the D9-branes of table 1 and ensures the Lorentz invariance of the fermionic subspaces. This will be made explicit in the next section.

5. Even and odd E-dualities

In the action formalism, toroidal compactifications are described by the action eq. (2.14) in the background metric g_{ab} and antisymmetric tensor b_{ab}. The left and right momenta of closed strings are given by eq. (2.15) which can be rewritten as

$$\begin{aligned} \mathbf{p_R} &= \frac{1}{2} m_a \mathbf{e^a} + n^a e_{ab} \mathbf{e^b} , \\ \mathbf{p_L} &= \mathbf{p_R} - 2n^a \mathbf{e_a} , \end{aligned} \tag{5.1}$$

where $\{\mathbf{e_a}\}$ defines the configuration space torus (see eq. (2.16)) and $\{\mathbf{e^a}\}$ is the lattice-dual basis. The metric g_{ab} is given by eq. (2.17) and we have defined

$$e_{ab} = g_{ab} + b_{ab} . \tag{5.2}$$

Here again, we concentrate on the SO(16) contribution of $E_8 \times$ SO(16). Therefore, from now on, the indices a, b run from 1 to 8. Indeed, the E_8 lattice partition function will not relevant, except for a subtlety in the open sector to be discussed at the end of this section.

In the case of EN lattices and in particular for the OB$_b$ and IIB$_b$ theories, there is a well-known lagrangian realization [19] which we shall refer to as the 'standard' realization. It is achieved by taking, in accordance with eq. (3.5), $\mathbf{w_a} = 2\mathbf{e_a} = \mathbf{r_a}$ with $\mathbf{r_a}$ the simple roots, and by choosing the following 'canonical' constant antisymmetric background field

$$b_{ab} = +\mathbf{e_a}.\mathbf{e_b} \quad \text{for} \quad a > b\,, \qquad = -\mathbf{e_a}.\mathbf{e_b} \quad \text{for} \quad a < b\,; \qquad b_{ab} = 0 \quad \text{for} \quad a = b\,. \tag{5.3}$$

This is not the only choice. Vectors $\mathbf{w_a}$ spanning different unit cells of the lattice along with canonical or non-canonical b_{ab} are also possible. Here we shall restrict, for the four bosonic theories, the lagrangian description to canonical b_{ab} tensors given by eqs. (5.3), whether or not $2\mathbf{e_a}$ is a root. In appendix B, we present explicitly such canonical realizations. The corresponding sets of basis vectors $\{\mathbf{w_a}\}_r$ $r = 1, \ldots, 4$ given there span the four sublattices (r) of eqs. (3.6). Note that for the two EN cases OB$_b$ and IIB$_b$, these realizations are not standard. Indeed the unit cell chosen is not spanned by simple roots and furthermore one of the vector is not a root and has $\mathbf{w^2} = 4$ (instead of $\mathbf{w^2} = 2$). This fact will be an important ingredient when solving the dilemma of the previous section.

Consider E-duality transformations [20]. On e_{ab}, it is defined as follows

$$e \to \frac{e^{-1}}{4} , \tag{5.4}$$

with

$$\left(\frac{e^{-1}}{4} \right)^{ab} = G^{ab} + B^{ab} . \tag{5.5}$$

When the transformation eqs. (5.4), (5.5) is performed, the hamiltonian of the closed string sector is invariant under the combined exchange of the background fields g_{ab} and b_{ab} with the E-dual metric G^{ab} and antisymmetric tensor B^{ab}, and of winding modes with momenta, $m_a \leftrightarrow n^a$.

In the open string sectors E-duality interchanges [21] Dirichlet boundary conditions given by eq. (3.1) (or expressed in the closed string tree channel by eq. (3.2)) with generalized Neumann boundary conditions defined by

$$[g_{ab}\partial_\sigma X^b - b_{ab}\partial_\tau X^b]|_{\sigma=0}^{\sigma=\pi} = 0\,. \tag{5.6}$$

In the tree channel this condition becomes

$$g_{ab}\partial_\tau X^b - b_{ab}\partial_\sigma X^b = 0\,, \tag{5.7}$$

or

$$m_a = 0\,. \tag{5.8}$$

Thus E-duality maps a D-brane localized on the torus onto a D-brane completely wrapped on it or vice-versa.

We now analyse the effect of the E-duality on the left and right momenta eq. (5.1). Let $\{\mathbf{E^a}\}$ be a basis of the E-dual lattice. We have $G^{ab} = \mathbf{E^a}.\mathbf{E^b}$. We denote by $\{\mathbf{E_a}\}$ its lattice-dual basis defined by $\mathbf{E_a}.\mathbf{E^b} = \delta_a^b$. E-duality transforms $\mathbf{p_R}$ and $\mathbf{p_L}$ according to

$$\begin{aligned} \mathbf{e_a} &\to \mathbf{E^a}\,, \\ \mathbf{e^a} &\to \mathbf{E_a}\,, \\ m_a &\leftrightarrow n^a\,. \end{aligned} \tag{5.9}$$

We may choose the basis $\mathbf{E_a}$ in such a way that the E-dual right momenta $\mathbf{p_R^D}$ are equal to $\mathbf{p_R}$ given by eq. (5.1). We get,

$$\mathbf{E_a} = 2e_{ab}\mathbf{e^b}. \tag{5.10}$$

Using eqs. (5.9) and (5.10) we have for the E-dual right and left momenta $\mathbf{p_R^D}$ and $\mathbf{p_L^D} = (\mathbf{p_L^D})_a\mathbf{e^a}$

$$\begin{aligned} \mathbf{p_R^D} &= \mathbf{p_R}\,, \\ (\mathbf{p_L^D})_a &= -(e^t.e^{-1})_a{}^b(\mathbf{p_L})_b\,, \end{aligned} \tag{5.11}$$

where the transpose matrix $e^t_{ab} = g_{ab} - b_{ab}$. The left momenta are rotated under E-duality [21, 22]. Indeed, the transformation eq. (5.11) is represented in the cartesian basis by a rotation matrix $\mathcal{R}_i{}^j$:

$$(\mathbf{p_L^D})_i = \mathcal{R}_i{}^j(\mathbf{p_L})_j \qquad \text{with} \quad \mathcal{R}_\mathrm{i}{}^\mathrm{j} = -\mathbf{e^a}{}_\mathrm{i}(\mathrm{e^t.e^{-1}})_\mathrm{a}{}^\mathrm{b}\mathbf{e_b}{}^\mathrm{j}\,. \tag{5.12}$$

The effect of this rotation is interesting. One might have thought that, at an enhanced symmetry point, E-duality always maps the theory onto itself. However this need not be the case if the closed string spectrum can be mapped onto a spectrum of a different theory with the same degeneracy. This is the case for the OA_b and OB_b spectra or for the IIA_b and IIB_b spectra. In fact, starting from any lagrangian realization of OA_b and IIA_b, E-duality *always* maps OA_b onto OB_b and IIA_b onto IIB_b. The demonstration of this theorem is presented in appendix C. We shall characterize such an E-duality as 'odd' to distinguish it from the 'even' E-dualities which map a theory onto itself. E-duality applied to OA_b and IIA_b theories is thus always odd.

On the other hand starting with OB_b or IIB_b one may have both even and odd E-dualities. For instance, OB_b and IIB_b in their standard lagrangian realization are mapped onto themselves by even E-duality, but the inverse of an E-duality mapping OA_b (resp. IIA_b) onto OB_b (resp. IIB_b) is of course odd. Realizations connected by odd (resp. even) E-dualities are called odd (resp. even). Examples of odd realizations are given in appendix B.

One can understand the origin of odd E-dualities in the following way. It is always possible to construct a unit cell of the OB_b or IIB_b lattice, which contains a single vector $\mathbf{w_a}$ of squared length 4 and to deduce from it a canonical lagrangian realization (see appendix B). Taking the determinant of eq. (5.10), we get the unit cell volume V_{dual} of the E-dual theory in terms of the unit cell volume V of the original theory

$$V_{\mathrm{dual}}^{-1} = 2^8(\det e)\; V^{-1}\,. \tag{5.13}$$

Since we are dealing with canonical realizations, e_{ab} is a triangular matrix. Henceforth,

$$\det e = \prod_{a=1}^{8} \mathbf{e_a}.\mathbf{e_a} = 2^{-16}\prod_{a=1}^{8} \mathbf{w_a}.\mathbf{w_a}\,, \tag{5.14}$$

where $\mathbf{w_a}$ define a unit cell of OB_b or IIB_b (modulo the 'dummy' E_8). If all the $\mathbf{w_a}$ are roots of SO(16) or E_8, we get $V_{\mathrm{dual}} = V$. On the other hand, if one takes a unit cell of the root lattice containing one vector of squared length four, we get $V_{\mathrm{dual}} = V/2$. This agrees with an E-duality transform of the volume of a unit cell from OB_b to that of OA_b, and similarly, from IIB_b to IIA_b. Such a construction of odd E-dualities is exhibited in appendix B.

We are now in a position to discuss the existence of the different bosonic Dp-branes given in table 4.

We take as input the bosonic D9-branes of table 1 which are uniquely defined independently of the lagrangian realization. The Dp-brane spectra are then fully determined by considering *odd* realizations of the four theories. The lower dimensional ones are univocally obtained by performing together odd E-dualities and T-dualities, as required by the Lorentz invariance of the truncated theories. Consider indeed the OA_b and IIA_b theories, which only admit odd realizations. Their amplitudes $\mathcal{A}_p$ for odd p are obtained from Dirichlet boundary conditions in the compact space while the amplitudes $\mathcal{A'}_p$ for even p describe D(p+8)-branes wrapped in the SO(16) compact space. The latter are obtained from an odd realization of OB_b and IIB_b by performing an E-duality on the Dp-branes of these two theories. Similarly, even p amplitudes of the OB_b and IIB_b theories are given by wrapped branes which are the E-dual of Dirichlet branes of OA_b and IIA_b.

The OB_b and IIB_b also have even realizations in which the Neumann boundary conditions in compact space simply duplicate the spectra of the Dirichlet conditions. Hence they would only generate the odd amplitudes $\mathcal{A}_p$ while the odd realizations generate all of them. The richer open spectra of the odd realizations point towards their fundamental character.

Up to now we have considered Dirichlet boundary conditions on the E_8 torus. One may also consider Neumann boundary conditions, that is D-branes wrapped on this torus. The corresponding tensors g_{ab} and b_{ab} are equivalent to those defined on the SO(16) torus for the IIB theory and thus compactifications on the E_8 torus admit even and odd realizations. Both have $\mathcal{G} = E_8 \times E_8$ symmetry in the closed string sectors but this symmetry is broken in odd realizations for open strings with Neumann boundary conditions. In fact the truncation of the amplitudes in this case is inconsistent, and for Neumann boundary conditions one must take an even realization. To understand this point, we first notice that the truncation of E_8 is of a different nature than the truncation of $\mathrm{SO}'(8)$. In the former case, one indeed drops, in addition to the oscillator states, *all* momentum states. Thus the only remnant of E_8 in the truncated theory is the coefficient multiplying the o_{16} character in the E_8 lattice contribution to the annulus amplitude. The latter appears, for Dirichlet boundary conditions, in the third row of table 1, both in the tree and in the loop expression for this amplitude. For even realizations of E_8, these correct coefficients are not altered when considering Neumann branes, as their lattice partition functions are the same as the Dirichlet ones. For odd realizations, the Neumann amplitude (or equivalently the Dirichlet amplitude of the E-dual theory) appears in the fourth row of the same table. We see immediately that, while the coefficient of the truncated theory in the loop expression of the annulus amplitude is correct, the coefficient is off by a factor of 2 when expressed in the tree channel. This translates the fact that for odd realizations, truncation of the E_8 contribution does not commute with the S-modular transformation in the open string sectors.

We see that, if we choose for all four theories the D9-branes to be those uniquely determined by the Dirichlet conditions in compact space, consistency of the truncation with Lorentz and modular invariance fully determines, for all lower dimensional fermionic branes, the amplitudes of the bosonic parents to be those given in table 4. The crucial element which led to this astonishing result, is the existence of odd E-dualities. These reflect a hidden link between the fermionic theories, expressed in the bosonic string through the global group properties of $\widetilde{\mathrm{SO}}(16)$.

6. Tadpole-free descendants and truncation

In this section we describe the open descendants of the closed bosonic theories. These are determined by imposing the *tadpole condition* [4] on the bosonic string, namely by imposing that divergences due to massless tadpoles cancel in the vacuum amplitudes. We will show that the bosonic OB_b, IIB_b and OA_b theories admit tadpole-free open bosonic descendants and that those descendants give after truncation the three open fermionic string theories which are anomaly or tadpole-free. In our previous paper [3], we already derived the open descendants of OB_b and IIB_b which, after truncation, give respectively the tadpole-free $[\mathrm{SO}(32-n)\times\mathrm{SO}(n)]^2$ theory and the anomaly free SO(32) type I theory. These results were derived for OB_b and IIB_b from the annulus $\mathcal{A}$ and Klein bottle $\mathcal{K}$ amplitudes describing wrapped D25-branes and orientifolds. We shall keep this approach here for the orientifolds, but in accordance with the method used in the previous sections, we consider Dirichlet D9-branes. This approach is equivalent to the one of reference [3] for the OB_b and IIB_b theories,[10] but will allow the construction of the open descendant of OA_b which was not obtained there.

A first step in obtaining the open descendant corresponding to the four bosonic string theories characterized by the tori amplitudes $\mathcal{T}$ eqs. (2.10)–(2.13) is the construction of the Klein bottle amplitudes $\mathcal{K}$. These are obtained from the amplitudes $\mathcal{T}/2+\mathcal{K}$, which are the torus closed string partition functions $\mathcal{T}$ with the projection operator $(1+\Omega)/2$ inserted, where Ω interchanges the left and right sectors: $\Omega|L,R>=|R,L>$. This can be done for OB_b, IIB_b and OA_b but not for IIA_b, because Ω in that case is not a symmetry of the theory. The IIA_b theory does not admit any open descendant. As a consistency check we note that the realization of OA_b is the only one presented in appendix B which does not fulfill the condition $4b_{ab}\in\mathcal{Z}$ ensuring that Ω is a symmetry of a toroidally compactified theory [23]. The projection on Ω eigenstates amounts to impose the condition

$$\mathbf{p_R}=\mathbf{p_L}\,, \tag{6.1}$$

on the closed string momenta eqs. (2.15). Acting with $\Omega/2$ on the three different tori eqs. (2.10)-eqs. (2.12), one finds the three Klein bottle amplitudes[11]

$$\mathcal{K}(\mathrm{OB}_b) = \frac{1}{2}(o_{16}+v_{16}+s_{16}+c_{16})\,, \tag{6.2}$$

$$\mathcal{K}(\mathrm{IIB}_b) = \frac{1}{2}(o_{16}+s_{16})\,, \tag{6.3}$$

$$\mathcal{K}(\mathrm{OA}_b) = \frac{1}{2}(o_{16}+v_{16})\,. \tag{6.4}$$

The two remaining amplitudes with vanishing Euler characteristic, the annulus $\mathcal{A}$ and the Möbius strip $\mathcal{M}$, determine the open string partition function. The annulus amplitudes of D9-branes with generic Chan-Paton multiplicities are given in eqs. (3.23), (3.25) and (3.24). For the unoriented string considered here, $n=\bar{n}$ and the amplitudes must be divided by two. Denoting these amplitudes by $\mathcal{A}^{un}$, we have

$$\begin{aligned}\mathcal{A}^{un}(\mathrm{OB}_b) &= \frac{1}{2}(n_o^2+n_v^2+n_s^2+n_c^2)\,o_{16}+(n_on_v+n_sn_c)\,v_{16}+\\ &\quad+(n_on_s+n_vn_c)\,s_{16}+(n_on_c+n_vn_s)c_{16}\,,\end{aligned} \tag{6.5}$$

$$\mathcal{A}^{un}(\mathrm{OA}_b) = \frac{1}{2}(n_o^2+n_s^2)\,(o_{16}+v_{16})+n_on_s\,(s_{16}+c_{16})\,, \tag{6.6}$$

$$\mathcal{A}^{un}(\mathrm{IIB}_b) = \frac{1}{2}(n_o^2+n_v^2)\,(o_{16}+s_{16})+n_on_v\,(v_{16}+c_{16})\,. \tag{6.7}$$

[10]The amplitudes $\mathcal{A}$ used in reference [3] for OB_b and IIB_b describing D25-branes have the same lattice contribution as the D9-brane amplitudes used here and they can be interpreted as the E-dual of the latter in an even realization of the OB_b and IIB_b theories.

[11]Recall that we display only the SO(16) contribution of the amplitudes.

To get the Möbius amplitudes $\mathcal{M}$ and to implement the tadpole condition we express the Klein bottle and annulus amplitudes eqs. (6.2)–(6.4) and eqs. (6.5)–(6.7) as closed string tree channel amplitudes. Using the S-transformation of the characters eq. (2.6) and the scaling of the moduli in the integration variables one finds

$$\begin{aligned}
\mathcal{K}_{\rm tree}({\rm OB}_b) &= 2^5\, o_{16}\,, &(6.8)\\
\mathcal{A}^{un}_{\rm tree}({\rm OB}_b) &= 2^{-7}\Big[(n_o+n_v+n_s+n_c)^2\, o_{16} + \\
&\quad + (n_o+n_v-n_s-n_c)^2\, v_{16} + \\
&\quad + (n_o-n_v+n_s-n_c)^2\, s_{16} + \\
&\quad + (n_o-n_v-n_s+n_c)c_{16}\Big]\,, &(6.9)\\
\mathcal{K}_{\rm tree}({\rm IIB}_b) &= 2^4\,(o_{16}+s_{16})\,, &(6.10)\\
\mathcal{A}^{un}_{\rm tree}({\rm IIB}_b) &= 2^{-6}[(n_o+n_v)^2\, o_{16} + (n_o-n_v)^2\, s_{16}]\,, &(6.11)\\
\mathcal{K}_{\rm tree}({\rm OA}_b) &= 2^4\,(o_{16}+v_{16})\,, &(6.12)\\
\mathcal{A}^{un}_{\rm tree}({\rm OA}_b) &= 2^{-6}[(n_o+n_s)^2\, o_{16} + (n_o-n_s)^2\, v_{16}]\,. &(6.13)
\end{aligned}$$

To obtain the Möbius amplitudes $\mathcal{M}_{\rm tree}$ from $K_{\rm tree}$ and $A^{un}_{\rm tree}$, one requires that each term in the power series expansion of the total tree channel amplitude $\mathcal{K}_{\rm tree}+\mathcal{A}_{\rm tree}+\mathcal{M}_{\rm tree}$ be a perfect square. One gets

$$\begin{aligned}
\mathcal{M}_{\rm tree}({\rm OB}_b) &= \epsilon_1\,(n_o+n_v+n_s+n_c)\,\hat{o}_{16}\,, &(6.14)\\
\mathcal{M}_{\rm tree}({\rm IIB}_b) &= \epsilon_2\,(n_o+n_v)\,\hat{o}_{16} + \epsilon_3\,(n_o-n_v)\,\hat{s}_{16}\,, &(6.15)\\
\mathcal{M}_{\rm tree}({\rm OA}_b) &= \epsilon_4\,(n_o+n_s)\,\hat{o}_{16} + \epsilon_5\,(n_o-n_s)\,\hat{v}_{16}\,, &(6.16)
\end{aligned}$$

where $\epsilon_i = \pm 1$ will be determined by tadpole conditions. The 'hat' notation in the amplitudes eqs. (6.14)–(6.16) means that the overall phase present in the characters $\hat{r}_{16}$ is dropped. This phase arises because the modulus over which $\mathcal{M}$ is integrated (and which is not displayed here) is not purely imaginary but is shifted by 1/2, inducing in the partition functions i_{16} an alternate shift of sign in its power series expansion as well as a global phase. This one half shift is needed to preserve the group invariance of the amplitudes [3]. A detailed discussion of the shift in general cases can be found in reference [4].

We now impose the tadpole conditions on the three theories, namely we impose the cancellation of the divergences due to the massless mode exchanges in the total amplitudes $\mathcal{K}_{\rm tree}+\mathcal{A}_{\rm tree}+\mathcal{M}_{\rm tree}$.

We briefly review the results for ${\rm OB}_b$ and ${\rm IIB}_b$ theories already given in detail in [3]. For ${\rm OB}_b$, inspecting eqs. (6.8), (6.9) and (6.14), we see that there are massless modes coming from the Dedekind functions (describing the non-compact space) and o_{16} at level one, and from s_{16} and c_{16} at level zero. Taking into account the alternating signs in the Möbius amplitudes, tadpole cancellation at level one implies $\epsilon_1 = 1$ and $n_o+n_v+n_s+n_c = 64$. Cancellation of the tadpole arising at level zero implies $n_o = n_v$ and $n_s = n_c$. Consequently the tadpole-free descendant of ${\rm OB}_b$ is characterized by a Chan-Paton group given by $[{\rm SO}(n)\times{\rm SO}(32-n)]^2$. For ${\rm IIB}_b$, eqs. (6.10), (6.11) and (6.15) yield tadpoles from the Dedekind functions and o_{16} at level one, and from s_{16} at level 0. The first tadpole condition gives $\epsilon_2 = +1$ and $n_o + n_v = 32$, the second gives $\epsilon_3 = -1$ and $n_o - n_v = 32$. Thus $n_v = 0$ and the open descendant of IIB_b has the Chan-Paton group SO(32). After truncation, the bosonic open descendant of IIB_b gives the anomaly-free type I theory and the bosonic descendant of OB_b gives the fermionic tadpole-free $[\mathrm{SO}(n) \times \mathrm{SO}(32-n)]^2$ theory [3].

We now turn to the construction of the descendant of OA_b which was not discussed in [3]. The eqs. (6.12), (6.13) and (6.16) yield only tadpoles from the Dedekind functions and o_{16} at level 1. The tadpole condition gives then $\epsilon_4 = +1$ and $n_o + n_s = 32$. There is no condition on ϵ_5. The Chan-Paton group is determined, as usual, by the massless vector contribution to the open string partition function $\mathcal{A} + \mathcal{M}$. There are $(1/2)(32^2 - 32)$ such vectors. Consequently the tadpole-free open descendant of OA_b has the Chan-Paton group $\mathrm{SO}(n) \times \mathrm{SO}(32-n)$. After truncation of this theory one recovers the open fermionic string theory with gauge group $\mathrm{SO}(n) \times \mathrm{SO}(32-n)$ which is the tadpole-free open descendant of the 10-dimensional type OA theory discussed in references [17, 24].

Starting with the bosonic tadpole-free open string theories and performing the universal truncation, we thus recover all the tadpole-free and anomaly-free open fermionic string theories.

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A. T-duality and the Lorentz algebra

To obtain a D8-brane from a fermionic D9-brane in the light cone gauge, or equivalently in the fermionic subspace of a bosonic string, one may compactify a non-compact direction, say the 8th, perform a T-duality and then decompactify. Compactification breaks the $\mathrm{SO}_{\mathrm{trans}}(8)$ to $\mathrm{SO}_{\mathrm{trans}}(7) \times \mathrm{U}(1)$ and the SO(8) generators L^{ab} $(a, b = 1, 2, \ldots, 8)$ decompose in SO(7) generators L^{ij} $(i, j = 1, 2, \ldots, 7)$ and a vector $L^{i\,8}$. One may then define separately the rotations $L^{i\,8}_L$ and $L^{i\,8}_R$ in the left and right closed string sectors. T-duality amounts to perform a parity operation in the left sector and hence sends $L^{i\,8}_L \to -L^{i\,8}_L$, $L^{i\,8}_R \to L^{i\,8}_R$. The light-cone so(8) algebra of the truncated theory was identified with the diagonal algebra $\mathrm{so}_{\mathrm{diag}}(8) = \mathrm{diag}[\mathrm{so}_{\mathrm{trans}}(8) \times \mathrm{so}_{\mathrm{int}}(8)]$. The generators J^{ab} of this algebra are given by

$$J^{ab} = L^{ab} + K^{ab}_o \tag{A.1}$$

where the operators K^{ab}_o belong to the internal $\mathrm{so}_{\mathrm{int}}(8)$ and are the zero-modes of the full $\widehat{\mathrm{so}}_{\mathrm{int}}(8)$ affine Lie algebra. Compactifying the 8th direction and performing a T-duality does not affect the right generators J^{i8}_R of $\mathrm{so}_{\mathrm{diag}}(8)$ associated with the 8th direction but implies that the left generators become $J_L^{i8} = -L_L^{i8} + K_{oL}^{i8}$. Inspecting the extension of $\mathrm{so}_{\mathrm{diag}}(8)$ to $\mathrm{so}_{\mathrm{diag}}(9,1)$ described in [2] one can see that the left algebra no longer closes. The remedy is to perform on the generator K_{oL}^{ij} of the Kac-Moody algebra, *together with* the T-duality, the transformation

$$\begin{aligned} K_{oL}^{ij} &\to K_{oL}^{ij} \qquad i,j = 1,\ldots,7 \\ K_{oL}^{i\,8} &\to -K_{oL}^{i\,8}\,. \end{aligned} \tag{A.2}$$

The transformations eq. (A.2) on the left $\widehat{\mathrm{so}}_{\mathrm{int}}(8)$ Kac-Moody generators implies that, when expressed in the Cartan-Weyl basis one has for the fourth Cartan generator H_{4L}[12]

$$H_{4L} \to -H_{4L}\,. \tag{A.3}$$

This implies a flip in sign of the fourth component of all the vectors of the left lattice as compared to the right lattice. This maps the $(o)_8$ and the $(v)_8$ left lattices onto themselves but interchanges the $(s)_8$ and $(c)_8$ lattices. Thus, as expected, to ensure closure of the left Lorentz algebra in the fermionic subspace, one must accompany the T-duality in the 8th dimension by the switch

$$\begin{aligned} \mathrm{OB} &\Leftrightarrow \mathrm{OA}\,, \\ \mathrm{IIB} &\Leftrightarrow \mathrm{IIA}\,. \end{aligned} \tag{A.4}$$

Hence, to ensure Lorentz invariance in the truncation, a corresponding switch must be made in the bosonic parents when a T-duality is performed in the bosonic theory.

B. Lagrangian realizations of the theories

We shall give explicit odd lagrangian realizations of the four bosonic theories which are related for OB_b, OA_b and IIB_b, IIA_b by odd E-dualities.

We first list basis vectors of a unit cell $\{2\,\mathbf{e_{ai}}\}$ and of the lattice-dual cell $\{\mathbf{e^a}_{\mathbf{i}}/2\}$ for the OB_b and OA_b theories and quote their volume.

The dual cells $\{\mathbf{e^a}_{\mathbf{i}}/2\}$ of OB_b and OA_b are respectively unit cells of the weight lattice $(o)+(v)+(s)+(c)$ and the lattice $(o)+(v)$ of SO(16), in accordance with the D9-brane tree amplitudes eqs. (3.3) given by $\mathbf{p_L} = \mathbf{p_R} = (1/2)m_a\mathbf{e^a}$. The spectra of D25-branes and of closed strings depend on b_{ab} and we now write the metric tensors $g_{ab} = \mathbf{e_a}\cdot\mathbf{e_b}$ and the canonical b_{ab} eq. (5.3) for both theories.

We can now write the E-dual basis $\{\mathbf{E^a}_{\mathbf{i}}\}$ of $\{\mathbf{e_{ai}}\}$ and its lattice-dual $\{\mathbf{E_{ai}}\}$ given by eqs. (5.2) and (5.10). The basis chosen for the unit OB_b cell contains a root lattice vector of squared length 4. As expected from the discussion in section 5, the realizations of OB_b and OA_b of table 5 are both odd. We indeed see by comparing table 5 with table 7 that, in the particular choice of unit cells taken here, $\{\mathbf{e_{ai}}\}_{\mathrm{OB}_b} = \{\mathbf{E^a}_{\mathbf{i}}\}_{\mathrm{OA}_b}$ and $\{\mathbf{e_{ai}}\}_{\mathrm{OA}_b} = \{\mathbf{E^a}_{\mathbf{i}}\}_{\mathrm{OB}_b}$. It is easily verified from these tables by rewriting eq. (2.15) as

$$\begin{aligned} \mathbf{p_R} &= \frac{1}{2}(m_a\mathbf{e^a} + n^a\mathbf{E_a})\,, \\ \mathbf{p_L} &= \mathbf{p_R} - 2n^a\mathbf{e_a}\ , \end{aligned} \tag{B.1}$$

[12]Here the four Cartan generators H_i, $i = 1,\ldots,4$ correspond to the 4 commuting rotations $K^{2i-1,2i}$.

OB_b	OA_b
$\{2\,\mathbf{e_{ai}}\}$	
$\begin{bmatrix} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&1\\ 0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&2 \end{bmatrix}$ $\mathrm{Det}\{2\,\mathbf{e_{ai}}\} = 2$	$\begin{bmatrix} 1&-1&0&0&0&0&0&0\\ 0&1&-1&0&0&0&0&0\\ 0&0&1&-1&0&0&0&0\\ 0&0&0&1&-1&0&0&0\\ 0&0&0&0&1&-1&0&0\\ 0&0&0&0&0&1&-1&0\\ 0&0&0&0&0&0&1&-1\\ 0&0&0&0&0&0&0&1 \end{bmatrix}$ $\mathrm{Det}\{2\,\mathbf{e_{ai}}\} = 1$
$\{\,\mathbf{e^a_i}/2\}$	
$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ \frac{-1}{2}&\frac{-1}{2}&\frac{-1}{2}&\frac{-1}{2}&\frac{-1}{2}&\frac{-1}{2}&\frac{-1}{2}&\frac{1}{2} \end{bmatrix}$ $\mathrm{Det}\{\,\mathbf{e^a_i}/2\} = 1/2$	$\begin{bmatrix} 1&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 1&1&1&1&0&0&0&0\\ 1&1&1&1&1&0&0&0\\ 1&1&1&1&1&1&0&0\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1 \end{bmatrix}$ $\mathrm{Det}\{\,\mathbf{e^a_i}/2\} = 1$

Table 5: Basis $\mathbf{e_a}$ and lattice-dual basis $\mathbf{e^a}$ for odd realizations of the OB_b and OA_b theories.

OB_b	OA_b
$\{4g_{ab}\}$	
$\begin{bmatrix} 2&1&1&1&1&1&1&2\\ 1&2&1&1&1&1&1&2\\ 1&1&2&1&1&1&1&2\\ 1&1&1&2&1&1&1&2\\ 1&1&1&1&2&1&1&2\\ 1&1&1&1&1&2&1&2\\ 1&1&1&1&1&1&2&2\\ 2&2&2&2&2&2&2&4 \end{bmatrix}$	$\begin{bmatrix} 2&-1&0&0&0&0&0&0\\ -1&2&-1&0&0&0&0&0\\ 0&-1&2&-1&0&0&0&0\\ 0&0&-1&2&-1&0&0&0\\ 0&0&0&-1&2&-1&0&0\\ 0&0&0&0&-1&2&-1&0\\ 0&0&0&0&0&-1&2&-1\\ 0&0&0&0&0&0&-1&1 \end{bmatrix}$
$\{4b_{ab}\}$	
$\begin{bmatrix} 0&-1&-1&-1&-1&-1&-1&-2\\ 1&0&-1&-1&-1&-1&-1&-2\\ 1&1&0&-1&-1&-1&-1&-2\\ 1&1&1&0&-1&-1&-1&-2\\ 1&1&1&1&0&-1&-1&-2\\ 1&1&1&1&1&0&-1&-2\\ 1&1&1&1&1&1&0&-2\\ 2&2&2&2&2&2&2&0 \end{bmatrix}$	$\begin{bmatrix} 0&1&0&0&0&0&0&0\\ -1&0&1&0&0&0&0&0\\ 0&-1&0&1&0&0&0&0\\ 0&0&-1&0&1&0&0&0\\ 0&0&0&-1&0&1&0&0\\ 0&0&0&0&-1&0&1&0\\ 0&0&0&0&0&-1&0&1\\ 0&0&0&0&0&0&-1&0 \end{bmatrix}$

Table 6: The g_{ab} and b_{ab} tensors for the odd OB_b and OA_b realizations presented in table 5.

that we indeed correctly get the closed string spectra of both theories. Finally, the tree channel amplitude of the D25-brane of each theory (given by $m_a = 0$) is nothing else than the D9 tree channel amplitude of the other theory, as it should be in odd realizations.

OB_b	OA_b
$\{\,\mathbf{E_{ai}}/2\}$	
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$
$\mathrm{Det}\{\,\mathbf{E_{ai}}/2\} = 1$	$\mathrm{Det}\{\,\mathbf{E_{ai}}/2\} = 1/2$
$\{2\,\mathbf{E^{a}}_{\mathbf{i}}\}$	
$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$
$\mathrm{Det}\{2\,\mathbf{E^{a}}_{\mathbf{i}}\} = 1$	$\mathrm{Det}\{2\,\mathbf{E^{a}}_{\mathbf{i}}\} = 2$

Table 7: The E-dual basis $\mathbf{E^a}$ and its lattice-dual $\mathbf{E_a}$ for the odd OB_b and OA_b realizations presented in table 5.

IIB_b	IIA_b
$\{\,\mathbf{E_{ai}}/2\}$	
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & -2 \\ -1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & 0 & -2 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \end{bmatrix}$
$\mathrm{Det}\{\,\mathbf{E_{ai}}/2\} = 2$	$\mathrm{Det}\{\,\mathbf{E_{ai}}/2\} = 1$
$\{2\,\mathbf{E^{a}}_{\mathbf{i}}\}$	
$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{3}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
$\mathrm{Det}\{2\,\mathbf{E^{a}}_{\mathbf{i}}\} = 1/2$	$\mathrm{Det}\{2\,\mathbf{E^{a}}_{\mathbf{i}}\} = 1$

Table 8: The E-dual basis $\mathbf{E^a}$ and its lattice-dual $\mathbf{E_a}$ for the odd IIB_b and IIA_b realizations presented in table 9.

IIB_b	IIA_b
$\{2\,\mathbf{e_{ai}}\}$	
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{3}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $\mathrm{Det}\{2\,\mathbf{e_{ai}}\} = 1$	$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $\mathrm{Det}\{2\,\mathbf{e_{ai}}\} = 1/2$
$\{\,\mathbf{e^a}_\mathbf{i}/2\}$	
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-3}{2} \end{bmatrix}$ $\mathrm{Det}\{\,\mathbf{e^a}_\mathbf{i}/2\} = 1$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 & 0 & 0 & -2 \\ -1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & 0 & -2 \\ -1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ $\mathrm{Det}\{\,\mathbf{e^a}_\mathbf{i}/2\} = 2$
$\{4g_{ab}\}$	
$\begin{bmatrix} 2 & 1 & -1 & 1 & -1 & 1 & -2 & 0 \\ 1 & 2 & -1 & 1 & -1 & 1 & -2 & 0 \\ -1 & -1 & 2 & -1 & 1 & -1 & 2 & -1 \\ 1 & 1 & -1 & 2 & -1 & 1 & -2 & 0 \\ -1 & -1 & 1 & -1 & 2 & -1 & 2 & -1 \\ 1 & 1 & -1 & 1 & -1 & 2 & -2 & 0 \\ -2 & -2 & 2 & -2 & 2 & -2 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & 1 & 1 & \frac{1}{2} & 2 \end{bmatrix}$

Table 9: Basis $\mathbf{e_a}$ and lattice-dual basis $\mathbf{e^a}$ for odd realizations of the IIB_b and IIA_b theories and the corresponding metric tensors.

We now present in table 9 the basis vectors of a unit cell $\{\mathbf{e_{ai}}\}$ and of the lattice-dual cell $\{\mathbf{e^{ai}}\}$ for the IIB_b and IIA_b theories. Their metric tensor g_{ab} is also displayed but the antisymmetric tensor b_{ab}, which has the canonical form, is omitted. Again we have chosen a unit cell of IIB_b containing a vector of length squared 4 and we have odd realizations. The E-dual bases are shown in table 8. As in the previous $\mathrm{OB}_b - \mathrm{OA}_b$ case we get, comparing table 9 with table 8, $\{\mathbf{e_{ai}}\}_{\mathrm{IIB}_b} = \{\mathbf{E^a}_\mathbf{i}\}_{\mathrm{IIA}_b}$ and $\{\mathbf{e_{ai}}\}_{\mathrm{IIA}_b} = \{\mathbf{E^a}_\mathbf{i}\}_{\mathrm{IIB}_b}$. One may check as previously that the closed string spectrum, as well as the D9 and D25-brane spectra characteristic of odd realizations, are correctly given.

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C. Properties of E-duality

Here we will prove that E-duality *always* maps

$$\begin{aligned} \mathrm{OA}_b &\to \mathrm{OB}_b\,, \\ \mathrm{IIA}_b &\to \mathrm{IIB}_b\,. \end{aligned} \tag{C.1}$$

It suffices to show that one can not have $\mathrm{OA}_b \to \mathrm{OA}_b$ and $\mathrm{IIA}_b \to \mathrm{IIA}_b$, as the only different theory whose spectrum matches the OA_b (IIA_b) theory is the OB_b (IIB_b) theory. To prove these statements we use an identity between $\{\mathbf{e_a}\}$ defined in eqs. (2.16)–(2.17) and $\{\mathbf{E_a}\}$ defined in eq. (5.10). One has

$$\mathbf{E_a}.\mathbf{e_b} + \mathbf{E_b}.\mathbf{e_a} = 4\,\mathbf{e_a}.\mathbf{e_b}\,. \tag{C.2}$$

It is convenient to work with the vectors $\mathbf{W_a} = (1/2)\,\mathbf{E_a}$ and $\mathbf{w_a} = 2\,\mathbf{e_a}$ which belong to some sublattices (r) of the SO(16) weight lattice (see eqs. (3.5) and (5.9)). One has

$$\mathbf{W_a}.\mathbf{w_b} + \mathbf{W_b}.\mathbf{w_a} = \mathbf{w_a}.\mathbf{w_b}\,. \tag{C.3}$$

(*a*) $\mathrm{OA}_b \neq> \mathrm{OA}_b$

If one assumes $\mathrm{OA}_b \to \mathrm{OA}_b$ then, using eqs. (3.5)–(3.6), the sublattice is $(o)+(v)$ and at least one $\mathbf{w_a}$ belongs to (v). For this $\mathbf{w_a}$, eq. (C.3) becomes, picking $a=b$,

$$\mathbf{W_a}.\mathbf{w_a} + \mathbf{W_a}.\mathbf{w_a} = \mathbf{w_a}.\mathbf{w_a}\,. \tag{C.4}$$

But, were OA_b the E-dual of OA_b, $\mathbf{W_a}$ would also belong to $(o)+(v)$. This is impossible because the l.h.s of eq. (C.4) would be even and the r.h.s would be odd.

(*b*) $\mathrm{IIA}_b \neq> \mathrm{IIA}_b$

If IIA_b would be mapped onto itself, the $\mathbf{W_a}$ would belong to the root lattice (o) as they are the lattice-dual of $\mathbf{W^a}$ which would form a basis of the SO(16) weight lattice. Introducing fundamental weights $\mathbf{v_a}$ dual to the simple roots $\mathbf{r^b}$ one writes

$$\begin{aligned} \mathbf{W_a} &= n_{ac}\mathbf{r^c}\,, \\ \mathbf{w_b} &= m_b^c\mathbf{v_c}\,, \end{aligned} \tag{C.5}$$

and hence

$$\mathbf{W_a}.\mathbf{w_b} = n_{ac}m_b^c \in \mathcal{Z} \qquad \forall a,b\,. \tag{C.6}$$

Using eqs. (C.3) and (C.6) we would then find that

$$\mathbf{w_a}.\mathbf{w_b} \in \mathcal{Z} \qquad \forall a,b\,. \tag{C.7}$$

But the vectors $\mathbf{w_a}$ form a basis of IIA_b which must at least contain a couple of weight vectors $(\mathbf{w_a},\mathbf{w_b}) = (v,s)$ or (v,c) or (c,s). For such couples, the scalar product in eq. (C.7) is half-integer.

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